مسابقة في مادة الرياضيات	الاسم:
المدة: ثلاث ساعات	الرقم:

ملاحظة: - يتكوّن هذا الامتحان من سبع مسائل، يجب اختيار **خمس مسائل** منها فقط.
- في حال الإجابة عن أكثر من خمس مسائل، عليك شطب الإجابات المتعلّقة بالمسألة التي لم تعد من ضمن اختيارك، لأنّ التصحيح سيقتصر على إجابات المسائل الخمس الأولى غير المشطوبة.
- يسمح باستعمال آلة حاسبة غير قابلة للبرمجة او اختزان المعلومات او رسم البيانات.
- يستطيع المرشّح الإجابة بالترتيب الذي يناسبه (دون الالتزام بترتيب المسائل الواردة في المسابقة).

I- Functions and Complex numbers (4 points)

In the table below, only one among the proposed answers to each question is correct.
Write the number of each question and give, **with justification**, the answer that corresponds to it.

Nº	Questions	Proposed answers		
		a	b	c
1	The domain of definition of the function f given by $f(x) = \ln(e^x - e^3)$ is	$]3, +\infty[$	$]-\infty, 3[\cup]3, +\infty[$	$]0, 3[\cup]3, +\infty[$
2	$\lim\limits_{x\to+\infty} \dfrac{\ln x}{e^{2x} + x} =$	$+\infty$	0	1
3	If x is a non-zero real number, then $\left\lvert\dfrac{1 + ix}{x + i}\right\rvert =$	1	$\dfrac{x+1}{x-1}$	$\dfrac{x^2+1}{x^2-1}$
4	The set of solutions of the inequality $e^{-2x} - 1 < 0$ is	$]0, +\infty[$	$]-\infty, +\infty[$	$]-\infty, 0[$
5	The number of solutions of the equation $e^{\ln(x+1)} = \ln(e^{x^2+x})$ is	0	1	2

II- Probability (4 points)

U and V are two urns.

- U contains 4 red balls and 2 black balls.
- V contains 3 red balls and 2 black balls.

Part A

A player rolls a fair 6-sided die numbered 1, 2, 3, 4, 5 and 6.

- If the die shows 5, the player selects randomly and simultaneously 3 balls from U.
- Otherwise, the player selects randomly and successively with replacement 3 balls from V.

Consider the following events:

A: "The die shows 5"
R: " The three selected balls are red"
M: "The three selected balls have the same color".

1) a) Calculate the probability P(R / A) and deduce that $P(R \cap A) = \dfrac{1}{30}$.

b) Verify that $P(R / \overline{A}) = \dfrac{27}{125}$ and calculate P(R).

2) a) Calculate P(M / A) and $P(M / \overline{A})$.

b) Deduce that $P(M) = \dfrac{4}{15}$.

3) Knowing that the three selected balls are of the same color, calculate the probability that the die does not show 5.

Part B

In this part, we select randomly **1** ball from U and **2** balls successively without replacement from V.
Calculate the probability of selecting exactly one red ball among the three selected balls.

III- Complex numbers (4 points)

The complex plane is referred to a direct orthonormal system $(O ; \vec{u}, \vec{v})$.

Consider the points A, M and M′ with affixes $z_A = -i$, $z_M = z$ and $z_{M'} = z'$ such that $z' = \dfrac{i}{\bar{z} - i}$ with $z \neq -i$.

1) Determine the exponential form of z' when $z = 1 - i$.
2) a) Show that $\overline{z'}(z + i) = -i$.

   b) Show that $OM' \times AM = 1$ and that $(\vec{u} ; \overrightarrow{AM}) - (\vec{u} ; \overrightarrow{OM'}) = \dfrac{-\pi}{2}$ $[2\pi]$.

   c) Show that: if M moves on the circle with center A and radius 2, then M′ moves on a circle to be determined.

   d) Show that: if M′ moves on the y-axis deprived of O, then M moves on a line to be determined.
3) Let $z = x + iy$ and $z' = x' + iy'$ where x, y, x′ and y′ are real numbers.

   a) Show that $x' = \dfrac{-y-1}{x^2 + (y+1)^2}$ and $y' = \dfrac{x}{x^2 + (y+1)^2}$.

   b) Show that: if M moves on the line with equation $y = -x - 1$ deprived of A, then M′ moves on a line whose equation is to be determined.

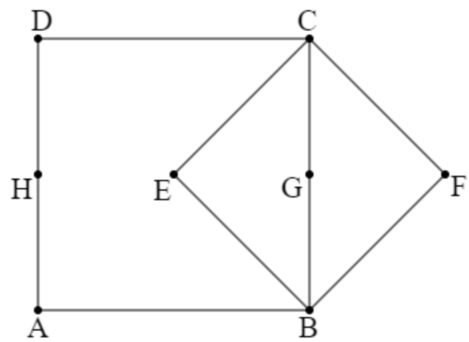
IV- Transformations (4 points)

In the adjacent figure, we have:

- ABCD and EBFC are two direct squares of centers E and G respectively.
- H is the midpoint of [AD].

Let S be the direct plane similitude with center I that transforms C onto B and D onto G.

$\alpha = -\dfrac{\pi}{2}$ is an angle of S and k is its ratio.



Part A

1) Verify that $k = \dfrac{1}{2}$.
2) Show that S(A) = E.
3) a) Show that the image of (CF) by S is (BF) and determine the image of (AD) by S.

   b) The two lines (AD) and (CF) intersect at point L. Show that S(L) = F.

   c) Deduce that ILF is a right triangle.
4) Let Q be the midpoint of [AB].

   a) Show that S(B) = Q.

   b) Show that the three points I, Q and C are collinear.

Part B

The plane is referred to a direct orthonormal system $(A ; \overrightarrow{AB}, \overrightarrow{AD})$.

1) Show that the complex form of S is $z' = -\dfrac{1}{2}iz + \dfrac{1}{2} + \dfrac{1}{2}i$.
2) Determine the algebraic form of the affix of the center I of S.

V- Functions (4 points)

Consider the function f defined over $\mathbb{R}$ as $f(x) = \frac{2}{1 - xe^{-x}}$ and denote by (C) its representative curve in an orthonormal system $(O ; \vec{i}, \vec{j})$.

1) Determine $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to +\infty} f(x)$. Deduce the two asymptotes to (C).

2) **a)** Show that $f'(x) = \frac{2e^{-x}(1 - x)}{(1 - xe^{-x})^2}$.

b) Deduce that f '(x) and (1 – x) have the same sign.

c) Set up the table of variations of f.

3) (C) has a point of inflection W(0 , 2).

Find an equation of (T), the tangent to (C) at the point W.

4) Draw (T) and (C).

5) Let h be the function given by $h(x) = \ln[(f(x) - 2)^2]$.

a) Determine the domain of definition of h.

b) Study the sense of variation of h on $]-\infty , 0[$.

VI- Functions (4 points)

Consider the function f defined on $]0 ; +\infty[$ as f(x) = x + lnx – ln(x + 1) and denote by (C) its representative curve in an orthonormal system $(O ; \vec{i}, \vec{j})$.

Let (d) be the line with equation y = x.

1) Determine $\lim_{\substack{x \to 0 \\ x > 0}} f(x)$. Deduce an asymptote to (C).

2) **a)** Show that $\lim_{x \to +\infty} f(x) = +\infty$.

b) Show that (d) is an asymptote to (C) at $+\infty$.

c) Show that (C) is below (d) for all $x \in]0 ; +\infty[$.

3) **a)** Verify that $f'(x) = 1 + \frac{1}{x(x+1)}$.

b) Set up the table of variations of f.

4) **a)** Show that the equation f(x) = 0 has a unique solution α.

b) Verify that 0.8 < α < 0.9.

c) The equation f(x) = 2 has a unique solution β. Show that α < β.

5) Draw (d) and (C).

6) Consider the function g given by $g(x) = \ln\left(\frac{f(x)}{f(x) - 2}\right)$.

Determine the domain of definition of g.

VII- Numerical sequences and integrals (4 points)

The following 4 parts are independent.

1) Consider the sequence (V_n) defined by $V_n = \int_2^3 e^{-x}(x-2)^n dx$ where n is an integer and $n \geq 1$. Show that (V_n) is a decreasing sequence.

2) Calculate the integral $\int (x^2 + x + 1)e^{-2x}dx$.

3) Consider the convergent sequence (U_n) defined by $U_0 = 3$ and $U_{n+1} = \frac{3U_n - 4}{U_n - 2}$ where $n \in \mathbb{N}$.

Knowing that $U_n > 2$ for all n, calculate the limit of (U_n).

4) The figure below shows the representative curves (C) and (C'), in an orthonormal system, of a function f and its derivative function f ' respectively.

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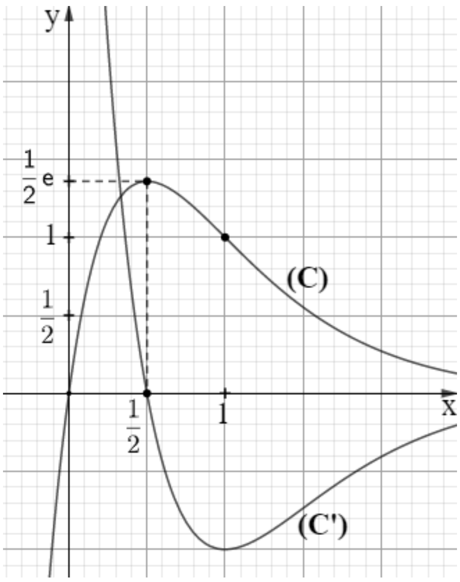


Calculate the area of the region bounded by (C'), the x-axis and the two lines of equations $x = \frac{1}{2}$ and $x = 1$.

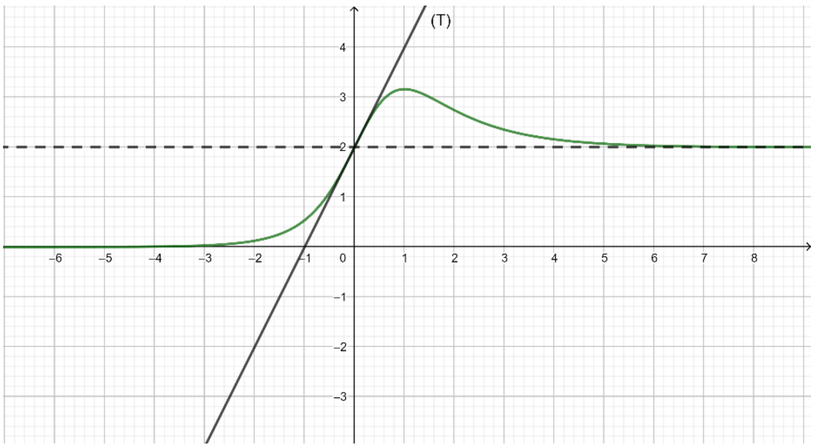
مشروع أسس التصحيح

	Question I	6 pts
1	$e^x - e^3 > 0$ $x > 3$ Answer: a	1.5
2	$\lim\limits_{x\to+\infty} \dfrac{\ln x}{e^{2x}+x} = \lim\limits_{x\to+\infty} \dfrac{\frac{1}{x}}{2e^{2x}+1} = \lim\limits_{x\to+\infty} \dfrac{1}{x(2e^{2x}+1)} = 0$ Answer: b	1.5
3	$\left\|\dfrac{1+ix}{x+i}\right\| = \dfrac{\sqrt{1+x^2}}{\sqrt{x^2+1}} = 1$ Answer: a	1
4	$e^{-2x} - 1 < 0$ Then $e^{-2x} - 1 < 0$ Then $e^{-2x} < 1$, then $-2x < 0$, then $x > 0$ Answer: a	1
5	$e^{\ln(x+1)} = \ln e^{x^2+x}$, condition $x > -1$ then $x+1 = x^2+x$, then $x^2 = 1$, then $x = -1$ (rejected) or $x = 1$ (accepted). Answer: b	1

	Question II	6 pts
A1a	$P(R / A) = \frac{C_4^3}{C_6^3} = \frac{1}{5}$ $P(R \cap A) = P(R / A) \times P(A) = \frac{1}{5} \times \frac{1}{6} = \frac{1}{30}$	1
A1b	$P(R / \overline{A}) = \frac{3^3}{5^3} = \frac{27}{125}$ $P(R) = P(R \cap A) + P(R \cap \overline{A}) = \frac{1}{30} + \frac{5}{6} \times \frac{27}{125} = \frac{16}{75}$	1
A2a	$P(M / A) = P(R / A) = \frac{1}{5}$ $P(M / \overline{A}) = \frac{3^3+2^3}{5^3} = \frac{7}{25}$	1
A2b	$P(M) = P(M \cap A) + P(M \cap \overline{A}) = \frac{1}{5} \times \frac{1}{6} + \frac{7}{25} \times \frac{5}{6} = \frac{4}{15}$	1
A3	$P(\overline{A} / M) = \frac{P(\overline{A} \cap M)}{P(M)} = \frac{\frac{7}{30}}{\frac{4}{15}} = \frac{7}{8}$	1
B	$P(\text{exactly one red ball}) = \frac{4}{6} \times \left(\frac{2}{5} \times \frac{1}{4}\right) + \frac{2}{6} \times \left(\frac{3}{5} \times \frac{2}{4}\right) \times 2 = \frac{4}{15}$	1

	Question III	6 pts
1	If $z = 1 - \mathrm{i}$, then $z' = \frac{\mathrm{i}}{1+\mathrm{i}-\mathrm{i}} = \mathrm{i} = e^{\mathrm{i}\frac{\pi}{2}}$	0.5
2a	$\overline{z'}(z+\mathrm{i}) = \frac{-\mathrm{i}}{z+\mathrm{i}}(z+\mathrm{i}) = -\mathrm{i}$	0.5
2b	$OM' \cdot AM = \lvert z'\rvert \cdot \lvert z+\mathrm{i}\rvert = \frac{1}{\lvert z+\mathrm{i}\rvert} \cdot \lvert z+\mathrm{i}\rvert = 1$ $(\vec{u};\overrightarrow{AM}) - (\vec{u};\overrightarrow{OM'}) = \arg(z+\mathrm{i}) - \arg(z') \ (2\pi)$ $= \arg(z+\mathrm{i}) - \arg(\mathrm{i}) + \arg(\bar{z} - \mathrm{i}) \ (2\pi)$ $= \arg(z+\mathrm{i}) - \arg(\mathrm{i}) - \arg(z+\mathrm{i}) \ (2\pi)$ $= -\arg(\mathrm{i}) \ (2\pi) = -\frac{\pi}{2} \ (2\pi)$	0.5 0.5
2c	$AM = 2$, then $OM' = \frac{1}{2}$, then M' moves on a circle of center O and radius $\frac{1}{2}$.	1
2d	$(\overrightarrow{OM'};\overrightarrow{AM}) = (\vec{u};\overrightarrow{AM}) - (\vec{u};\overrightarrow{OM'}) = -\frac{\pi}{2} \ (2\pi)$, so $(AM) \perp (OM')$. Thus, if M' moves on y-axis deprived from O, then M moves on a line parallel to the x-axis and passing through A deprived from the point A.	1
3a	$x' = \frac{-y-1}{x^2+(y+1)^2}$ and $y' = \frac{x}{x^2+(y+1)^2}$	1
3b	If $y = -x-1$, then $x' = \frac{x}{2x^2} = \frac{1}{2x}$ and $y' = \frac{x}{2x^2} = \frac{1}{2x}$, then $y' = x'$. Therefore, M' moves on the line of equation $y = x$ deprived from the point O.	1

	Question IV	6 pts
A1	$k = \frac{BG}{CD} = \frac{\frac{1}{2}BC}{CD} = \frac{1}{2}$	0.5
A2	Method 1: ADC is a direct right isosceles triangle at D, so S(ADC) is a direct right isosceles triangle at S(D) = G, that is triangle EGB. Also S(C) = B, hence S(A) = E. Method 2 : $\frac{EG}{AD} = \frac{1}{2}$ and $(\overrightarrow{AD};\overrightarrow{EG}) = -\frac{\pi}{2} \ (2\pi)$ and S(D) = G, thus S(A) = E.	1
A3a	(CF) passes through C, so S(CF) passes through S(C) = B and is perpendicular to (CF), thus S(CF) = (BF). (AD) passes through D, so S(AD) passes through S(D) = G and is perpendicular to (AD), thus S(AD) = (GE).	1
A3b	$(AD) \cap (CF) = \{L\}$ and $(GE) \cap (BF) = \{F\}$ and S(I) = I, so S(L) = F.	0.5
A3c	Since S(L) = F, then $(\overrightarrow{IL};\overrightarrow{IF}) = -\frac{\pi}{2} \ (2\pi)$, then the triangle ILF is right angled at I.	0.5
A4a	ABCD is a direct square, so S(B) is the 4th vertex of the direct square BGEQ, so S(B) = Q.	0.5
A4b	$S \circ S$ is a dilation of ratio $-\frac{1}{4}$ (because $-\frac{\pi}{2} - \frac{\pi}{2} = -\pi$) $S \circ S(C) = S(S(C)) = S(B) = Q$, so I, Q and C are collinear.	1
B1	B(1) and C(1+i). $z' = \frac{1}{2}e^{\frac{-\pi}{2}\mathrm{i}}z + b$ and S (C) = B, thus $1 = \frac{1}{2}e^{\frac{-\pi}{2}\mathrm{i}}(1+\mathrm{i}) + b$, then $b = \frac{1}{2} + \frac{1}{2}\mathrm{i}$ Thus, $z' = \frac{-1}{2}\mathrm{i}z + \frac{1}{2} + \frac{1}{2}\mathrm{i}$	0.5
B2	$z_I = \frac{\frac{1}{2}+\frac{1}{2}\mathrm{i}}{1-\frac{-1}{2}\mathrm{i}} = \frac{3}{5} + \frac{1}{5}\mathrm{i}$.	0.5

<table>
<tr><td></td><td>Question V</td><td>6 pts</td></tr>
<tr><td>1</td><td>$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \frac{2}{1-xe^{-x}} = 0$
$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \frac{2}{1-xe^{-x}} = \frac{2}{1-0} = 2$
$y = 0$ and $y = 2$ are the two asymptotes</td><td>1</td></tr>
<tr><td>2a</td><td>$f'(x) = \frac{2e^{-x}(1-x)}{(1-xe^{-x})^2}$</td><td>0.5</td></tr>
<tr><td>2b</td><td>Since $2e^{-x} > 0$ and $(1-xe^{-x})^2 > 0$, then $f'(x)$ and $(1-x)$ have the same sign.</td><td>0.5</td></tr>
<tr><td>2c</td><td><table>
<tr><td>x</td><td>$-\infty$</td><td>1</td><td>$+\infty$</td></tr>
<tr><td>f′(x)</td><td>+</td><td>0</td><td>−</td></tr>
<tr><td>f(x)</td><td>0 ↗</td><td>3.2</td><td>↘ 2</td></tr>
</table></td><td>1</td></tr>
<tr><td>3</td><td>(T): $y = 2x + 2$</td><td>0.5</td></tr>
<tr><td>4</td><td>
</td><td>1.25</td></tr>
<tr><td>5a</td><td>$f(x) - 2 \neq 0$, then $f(x) \neq 2$, then $x \neq 0$</td><td>0.75</td></tr>
<tr><td>5b</td><td>$h'(x) = \frac{2(f(x)-2)f'(x)}{(f(x)-2)^2} < 0$ for all $x \in]-\infty; 0[$.
So h is decreasing over $]-\infty; 0[$.</td><td>0.5</td></tr>
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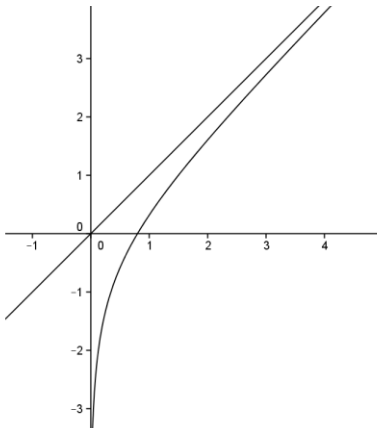
<table>
<tr><th></th><th>Question VI</th><th>6 pts</th></tr>
<tr><td>1</td><td>$\lim\limits_{x\to 0^+} f(x) = -\infty$
x = 0 is a vertical asymptote</td><td>0.5</td></tr>
<tr><td>2a</td><td>$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\left[x + \ln\frac{x}{x+1}\right] = +\infty + \ln 1 = +\infty + 0 = +\infty$</td><td>0.5</td></tr>
<tr><td>2b</td><td>$\lim\limits_{x\to+\infty}[f(x) - x] = \lim\limits_{x\to+\infty} \ln\frac{x}{x+1} = \ln 1 = 0$
So, $y = x$ is an O.A. at $+\infty$</td><td>0.25</td></tr>
<tr><td>2c</td><td>$f(x) - x = \ln x - \ln(x+1) < 0$ because $x < x+1$
So, (C) is below (d).</td><td>0.5</td></tr>
<tr><td>3a</td><td>$f'(x) = 1 + \frac{1}{x} - \frac{1}{x+1} = 1 + \frac{1}{x(x+1)}$</td><td>0.25</td></tr>
<tr><td>3b</td><td>Since x > 0 and x+1 > 0, then $f'(x) > 0$.
<table><tr><td>x</td><td>0 $+\infty$</td></tr><tr><td>$f'(x)$</td><td>+</td></tr><tr><td>f(x)</td><td>$-\infty$ ↗ $+\infty$</td></tr></table></td><td>0.5</td></tr>
<tr><td>4a</td><td>If $x \in]0;+\infty[$, f is defined, continuous and strictly increasing from $-\infty$ to $+\infty$, so the equation $f(x) = 0$ admits a unique root α.
$f(0.8) < 0$ and $f(0.9) > 0$, then $0.8 < \alpha < 0.9$</td><td>1</td></tr>
<tr><td>4b</td><td>Given the equation $f(x) = 2$ admits a unique root β.
$f(\alpha) = 0$, $f(\beta) = 2$ and $0 < 2$ and f is strictly increasing over $]0;+\infty[$, so $\alpha < \beta$.</td><td>0.5</td></tr>
<tr><td>5</td><td></td><td>1</td></tr>
<tr><td>6</td><td>$\frac{f(x)}{f(x)-2} > 0$
<table><tr><td>x</td><td>0</td><td>α</td><td>β</td><td>$+\infty$</td></tr><tr><td>f(x)</td><td>−</td><td>0 +</td><td>+</td><td></td></tr><tr><td>$f(x)-2$</td><td>−</td><td>−</td><td>0 +</td><td></td></tr><tr><td>$\frac{f(x)}{f(x)-2}$</td><td>+</td><td>−</td><td>+</td><td></td></tr></table>
So, $D =]0;\alpha[\cup]\beta;+\infty[$</td><td>0.5</td></tr>
</table>

	Question VII	6 pts
1	$V_{n+1} - V_n = \int_2^3 [e^{-x}(x-2)^{n+1} - e^{-x}(x-2)^n]dx$ $= \int_2^3 e^{-x}(x-2)^n(x-3)dx \le 0$ (because $2 \le x \le 3$, then $x-2 \ge 0$ and $x-3 \le 0$). Therefore, (V_n) is decreasing.	1.5
2	$\int (x^2+x+1)e^{-2x}dx = -\left(\frac{1}{2}x^2+x+1\right)e^{-2x} + C$	1.5
3	$U_{n+1} = \frac{3U_n-4}{U_n-2}$; $L = \lim\limits_{n\to+\infty} U_n$. $L = \frac{3L-4}{L-2}$, then $L^2 - 5L + 4 = 0$, then $L = 1$ (rejected) or $L = 4$ (accepted)	1.5
4	Area $= \int_{\frac{1}{2}}^{1} -f'(x)dx = f\left(\frac{1}{2}\right) - f(1) = \left(\frac{1}{2}e - 1\right)\ u^2$	1.5