عدد المسائل: خمس

مسابقة في مادة الرياضيات
المدة: ثلاث ساعات

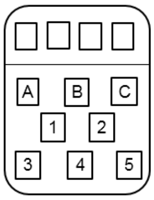
الاسم:
الرقم:

ملاحظة: - يسمح باستعمال آلة حاسبة غير قابلة للبرمجة او اختزان المعلومات او رسم البيانات.
- يستطيع المرشّح الإجابة بالترتيب الذي يناسبه (دون الالتزام بترتيب المسائل الواردة في المسابقة).

I- (2 points)

In the table below, only one among the proposed answers to each question is correct.

Write the number of each question and give, **with justification**, the answer that corresponds to it.

№	Questions	Proposed Answers		
		a	b	c
1	Consider the function f given by $f(x)=\ln\left(\frac{e^x}{e^x-2}\right)$. The domain of definition of f is	$]\ln 2\,;\,+\infty[$	$]0\,;\,+\infty[$	$]-\infty\,;+\infty[$
2	For all real numbers x, $\frac{e^{-x}}{e^{-x}+2}$ is equal to	$\frac{1}{3}$	$\frac{1}{1+2e^x}$	$\frac{-e^x}{-e^x+2}$
3	The equation $\ln^2x+\ln x-6=0$ has two roots x_1 and x_2. The product $x_1.x_2$ is equal to	-6	e^{-1}	e^{30}
4	A security entrance keyboard of a building is formed of three letters A, B and C and five digits 1, 2, 3, 4 and 5.  The entrance code is formed of one letter followed by a number consisting of three distinct digits. The number of all possible codes is	15	180	375

II- (3 points)

The complex plane is referred to a direct orthonormal system $(O; \vec{u}, \vec{v})$.

Consider the points A, M and M ' with affixes $z_A = -i$, $z_M = z$ and $z_{M'} = z'$ such that

$z' = \dfrac{z+i}{i\bar{z}}$ with $z \neq 0$.

Suppose that $z = x + iy$ and $z' = x' + iy'$ where x, y, x' and y' are real numbers.

1) Write z' in exponential form in the case where $z = e^{i\frac{\pi}{2}}$.
2) Write z in algebraic form in the case where $z' = z$.
3) a- Show that $OM' = \dfrac{AM}{OM}$.

   b- Show that when M varies on the perpendicular bisector of [OA], the point M′ varies on a circle (C) whose center and radius are to be determined.
4) In this part $x > 0$ and $y > 0$.

   a- Show that $\dfrac{z'+i}{z} = \dfrac{2y+1}{x^2+y^2}$ and deduce that (OM) and (M′A) are parallel.

   b- Show that $z' - z = \dfrac{i + z - iz\bar{z}}{i\bar{z}}$ and deduce that if M belongs to (C), then $MM' = OA$.

III- (3 points)

An urn U contains red balls and black balls holding distinct natural numbers.

- 60 % of the balls are red of which 80 % hold odd numbers.
- 70 % of the black balls hold odd numbers.

Part A

One ball is selected from the urn. Consider the following events:

R: “the selected ball is red”

O: “the selected ball holds an odd number”.

1) Show that the probability $P(O \cap R)$ is equal to 0.48 and calculate $P(O \cap \overline{R})$.
2) Deduce that $P(O) = 0.76$.
3) Are the events R and O independent? Justify your answer.

Part B

Suppose in this part that the number of balls in the urn U is 50.

1) Show that the number of red balls holding odd numbers is equal to 24.
2) Copy and complete the following table :

	Red	Black	Total
Odd			38
Even			
Total	30		50

3) Three balls are selected randomly and simultaneously from the urn U.

   a- Calculate the probability of selecting at least one red ball holding an odd number.

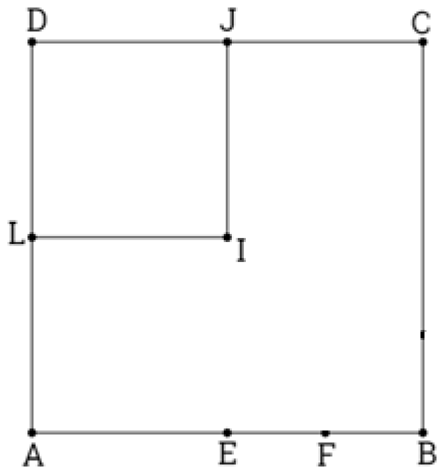
   b- The even numbered balls hold the numbers 2, 4, 6, …, 24.

   Knowing that the three selected balls hold even numbers, calculate the probability that the sum of the numbers on the balls is greater than 13.

IV- (4 points)

In the figure below:

- ABCD is a direct square with center I and side 8.
- E is the midpoint of [AB].
- F is the midpoint of [EB].
- J is the midpoint of [DC].
- L is the midpoint of [DA].



1) Let S be the direct plane similitude that maps F onto I and maps B onto J.

a- Show that $k = 2$ and $\alpha = \frac{\pi}{2}$ are respectively the ratio and an angle of S.

b- Show that E is the center of S.

2) Let S' be the direct plane similitude with ratio $k' = 2$ and an angle $-\frac{\pi}{2}$ that maps I onto B.

a- Show that S'(L) = C.

b- Show that the image of (LD) by S' is the line (DC).

c- Determine the image of (IC) by S'.

d- Determine the image of A by S'.

3) Let h = S o S'.

a- Show that h is a dilation whose ratio is to be determined.

b- Verify that $h(I) = J$.

c- Let W be the center of h, prove that W is the center of gravity of triangle ABJ.

V- (8 points)

Consider the two functions f and g defined over $]0;+\infty[$ as $f(x) = x^2 + 1 - \ln x$ and $g(x) = x + \frac{\ln x}{x}$.

Denote by (C) the representative curve of g in an orthonormal system $(O;\vec{i},\vec{j})$.

Let(d) be the line with equation $y = x$.

1) The table below is the table of variations of f.

x	0		$\frac{\sqrt{2}}{2}$		$+\infty$
$f'(x)$	‖	$-$	0	$+$	
$f(x)$	‖ $+\infty$	↘	$f(\frac{\sqrt{2}}{2})$	↗	$+\infty$

Show that $f(x) > 0$ for all $x > 0$.

2) a- Determine $\lim\limits_{x \to 0} g(x)$ and deduce an asymptote to (C).

b- Determine $\lim\limits_{x \to +\infty} g(x)$ and deduce that (d) is an asymptote to (C).

c- Study, according to the values of x, the relative positions of (C) and (d).

3) a- Show that $g'(x) = \frac{f(x)}{x^2}$ and deduce that $g'(x) > 0$ for all $x > 0$.

b- Set up the table of variations of g.

4) Prove that the equation $g(x) = 0$ has a unique root α and that $0.6 < \alpha < 0.7$.

5) Draw (d) and (C).

6) Calculate, in terms of α, the area of the region bounded by (C), (d) and the lines with equations $x = \alpha$ and $x = e$.

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امتحانات الشهادة الثانوية العامة
فرع: العلوم العامة

دورة العام ٢٠٢١ الاستثنائية
السبت في ٤ أيلول ٢٠٢١

أسس تصحيح مسابقة الرياضيات

I	Answers	3pts
1	$e^x - 2 > 0$; $e^x > 2$; $x > \ln 2$; $D_f =]\ln 2 ; +\infty[$ **(a)**	0.75
2	$\frac{e^{-x}}{e^{-x}+2} = \frac{e^{-x}}{e^{-x}+2} \times \frac{e^x}{e^x} = \frac{1}{1+2e^x}$ **(b)**	0.75
3	The roots of the equation $\ln^2 x + \ln x - 6 = 0$ are $x_1 = e^{-3}$ and $x_2 = e^2$. Then $x_1.x_2 = e^{-1}$ **(b)**	0.75
4	The number of all possible codes is $3 \times A_5^3 = 180$ **(b)**	0.75

II	Answers	4.5pts
1	$z' = 2i = 2e^{i\frac{\pi}{2}}$	0.5
2	$z' = z$; $iz\bar{z} = z + i$ $i(x^2 + y^2) = x + iy + i$; $x = 0$ and $x^2 + y^2 = y + 1$; $y^2 - y - 1 = 0$ $z_1 = \frac{1-\sqrt{5}}{2}i$ or $z_2 = \frac{1+\sqrt{5}}{2}i$	0.5
3a	$z' = \frac{z+i}{i\bar{z}}$; $\lvert z' \rvert = \frac{\lvert z+i \rvert}{\lvert i\bar{z} \rvert}$; $\lvert z' \rvert = \frac{\lvert z+i \rvert}{\lvert i \rvert \times \lvert \bar{z} \rvert}$; $\lvert z' \rvert = \frac{\lvert z+i \rvert}{\lvert z \rvert}$; $OM' = \frac{AM}{OM}$	0.5
3b	$MO = MA$; $OM' = 1$; M' moves on the circle (C) with center O and radius 1.	1
4a	$\frac{z'+i}{z} = \frac{\frac{z+i}{i\bar{z}}+i}{z} = \frac{z+i+i^2\bar{z}}{iz\bar{z}} = \frac{2iy+i}{i(x^2+y^2)} = \frac{2y+1}{x^2+y^2}$ $\frac{z'+i}{z} = \frac{2y+1}{x^2+y^2} \in \mathbb{R}$, then (OM) and (M 'A) are parallel.	1
4b	$z' - z = \frac{z+i}{i\bar{z}} - z = \frac{i + z - i z\bar{z}}{i\,\bar{z}}$; $M \in (C)$ then $\lvert z \rvert = \lvert \bar{z} \rvert = 1$ $\lvert z' - z \rvert = \lvert \frac{i + z - i z\bar{z}}{i\,\bar{z}} \rvert = \frac{\lvert i + z - i z\bar{z} \rvert}{\lvert i\,\bar{z} \rvert} = \frac{\lvert i + z - i z\bar{z} \rvert}{\lvert i \rvert \times \lvert \bar{z} \rvert} = \lvert i + z - i \times 1 \rvert = \lvert z \rvert = 1.$ And $OA = 1$ then $MM' = OA$.	1

<table>
<tr><th>III</th><th>Answers</th><th>4.5pts</th></tr>
<tr><td>A1</td><td>$P(I \cap R) = P(I / R) \times P(R) = 0.8 \times 0.6 = 0.48$
$P(I \cap \overline{R}) = P(I / \overline{R}) \times P(\overline{R}) = 0.7 \times 0.4 = 0.28$</td><td>1</td></tr>
<tr><td>A2</td><td>$P(I) = P(I \cap R) + P(I \cap \overline{R}) = 0.48 + 0.28 = 0.76$</td><td>0.5</td></tr>
<tr><td>A3</td><td>$P(I \cap R) = 0.48 \neq P(I) \times P(R) = 0.76 \times 0.6 = 0.456$
Then the events R and I are not independent.</td><td>0.5</td></tr>
<tr><td>B1</td><td>The number of red balls holding odd numbers is $50 \times 0.48 = 24$</td><td>0.5</td></tr>
<tr><td>B2</td><td><table>
<tr><th></th><th>Red</th><th>Black</th><th>Total</th></tr>
<tr><td>Odd</td><td>24</td><td>14</td><td>38</td></tr>
<tr><td>Even</td><td>6</td><td>6</td><td>12</td></tr>
<tr><td>Total</td><td>30</td><td>20</td><td>50</td></tr>
</table></td><td>0.5</td></tr>
<tr><td>B3.a</td><td>P (selecting at least one red ball holding an odd number) = $1 - \frac{C_{26}^3}{C_{50}^3} = \frac{85}{98}$</td><td>0.5</td></tr>
<tr><td>B3.b</td><td><u>2</u> ; <u>4</u> ; <u>6</u> ; 8 ; 10 ; 12 ; 14 ; 16 ; 18 ; 20 ; 22 ; 24
P (the sum of the numbers on the balls is greater than 13 / even) = $1 - \frac{C_3^3}{C_{12}^3} = \frac{219}{220}$
or $1 - \frac{C_1^1 \times C_1^1 \times C_1^1}{C_{12}^3} = \frac{219}{220}$</td><td>1</td></tr>
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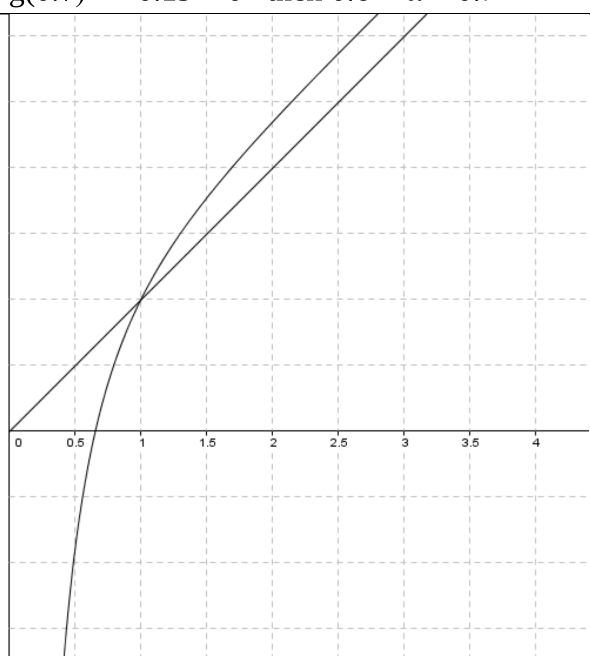
<table>
<tr><th>IV</th><th>Answers</th><th>6pts</th></tr>
<tr><td>1.a</td><td>S(F) = I and S(B) = J then $K = \frac{IJ}{FB} = \frac{4}{2} = 2$.
$\alpha = (\overrightarrow{FB}\,;\,\overrightarrow{IJ}) = (\overrightarrow{FB}\,;\,\overrightarrow{BC}) = \frac{\pi}{2} + 2k\pi\,;\, k \in \mathbb{Z}$.</td><td>1</td></tr>
<tr><td>1.b</td><td>E is symmetric of B with respect to F, then S(E) is symmetric of S(B) = J with respect to S(F) = I, then S(E) = E, then E is the center of S
EF = 2 and EI = 4 then EI = 2EF and $(\overrightarrow{EF}\,;\,\overrightarrow{EI}) = \frac{\pi}{2} + 2k\pi$ and since S(F) = I
then S(E) = E.</td><td>1</td></tr>
<tr><td>2.a</td><td>$\frac{BC}{IL} = \frac{4}{2} = 2 = K'$ and $(\overrightarrow{IL}\,;\,\overrightarrow{BC}) = (\overrightarrow{IL}\,;\,\overrightarrow{IJ}) = -\frac{\pi}{2} = \alpha'$ and $S'(I) = B$. Then $S'(L) = C$</td><td>0.5</td></tr>
<tr><td>2.b</td><td>$S'((LD))$ is a line ⊥ (LD) and passing through S'(L) = C. Then $S'((LD)) = (DC)$.</td><td>0.5</td></tr>
<tr><td>2.c</td><td>$S'((IC))$ is a line ⊥ (IC) and passing through S'(I) = B. Then $S'((IC)) = (BD)$.</td><td>0.5</td></tr>
<tr><td>2.d</td><td>$(IC) \cap (LD) = \{A\}$ then $\{S'(A)\} = S'((IC)) \cap S'((LD)) = (BD) \cap (DC) = \{D\}$.</td><td>0.5</td></tr>
<tr><td>3.a</td><td>$\alpha + \alpha' = \frac{\pi}{2} + \frac{-\pi}{2} = 0$ and k.k' = 4 then H = S o S' is a dilation of ratio 4.</td><td>0.5</td></tr>
<tr><td>3.b</td><td>h(I) = S o S'(I) = S(B) = J.</td><td>0.5</td></tr>
<tr><td>3.c</td><td>h(I) = J and W is the center of h, then $\overrightarrow{WJ} = 4\,\overrightarrow{WI}$
then $\overrightarrow{WJ} = 4(\overrightarrow{JI} - \overrightarrow{JW})$; $-3\overrightarrow{WJ} = 4 \times (\frac{1}{2}\overrightarrow{EJ})$ then $\overrightarrow{JW} = \frac{2}{3}\,\overrightarrow{JE}$ and E midpoint of [AB], then
W is the center of gravity of triangle JAB.</td><td>1</td></tr>
</table>

III	Answers	12pts
1	$f\left(\frac{\sqrt{2}}{2}\right) = \left(\frac{\sqrt{2}}{2}\right)^2 + 1 - \ln\frac{\sqrt{2}}{2} = \frac{3}{2} - \frac{\ln 2}{4} \approx 1.32 > 0$ then $f(x) > 0$ for all $x > 0$	**1**
2.a	$\lim_{x\to 0} g(x) = 0 + (-\infty) = -\infty$ then x = 0 is an asymptote to (C).	**1**
2.b	$\lim_{x\to +\infty} g(x) = (+\infty) + 0 = +\infty$ $g(x) - x = \frac{\ln x}{x}$; then $\lim_{x\to +\infty} [\,g(x) - x] = \lim_{x\to +\infty} \frac{\ln x}{x} = 0$; then (d) : y = x is an asymptote to (C).	**1**
2.c	$g(x) - x = \frac{\ln x}{x}$ $g(x) - x > 0$; $x > 1$; (C) is above to (d) $g(x) - x < 0$; $0 < x < 1$; (C) is below to (d) $g(x) - x = 0$; $x = 1$; (C) intersects (d) in (1 ; 1)	**1**
3.a	$g'(x) = 1 + \frac{(\ln x)'.x - 1.\ln x}{x^2} = \frac{x^2+1-\ln x}{x^2} = \frac{f(x)}{x^2}$; f(x) and g'(x) have the same sign ; $g'(x) > 0$	**1.5**
3.b	x: 0 → $+\infty$ $f'(x)$: + f(x): $-\infty$ ↗ $+\infty$	**1.5**
4	over] 0 ; +∞ [: g is continuous, and strictly increasing from –∞ to +∞ then the equation g(x) = 0 has a unique solution α. $g(0.6) \approx -0.25 < 0$ $g(0.7) \approx +0.19 > 0$ then $0.6 < \alpha < 0.7$	**1.5**
5		**2**
6	$A = \int_\alpha^1 [x - g(x)]dx + \int_1^e [g(x) - x]dx = -\frac{\ln^2 x}{2}]_\alpha^1 + \frac{\ln^2 x}{2}]_1^e = \frac{\alpha^4+1}{2}$ units of area.	**1.5**