عدد المسائل: ست	مسابقة في مادة الرياضيات المدة: أربع ساعات	الاسم: الرقم:

ملاحظة: - يسمح باستعمال آلة حاسبة غير قابلة للبرمجة او اختزان المعلومات او رسم البيانات.
- يستطيع المرشّح الإجابة بالترتيب الذي يناسبه (دون الالتزام بترتيب المسائل الواردة في المسابقة).

I- (2 points)

The complex plane is referred to a direct orthonormal system $\left(O\,;\,\vec{u}\,,\vec{v}\right)$.

A and B are two points with respective affixes $z_A = -4$ and $z_B = 2$.

M and M' are two points with respective affixes z and z' such that $z' = \dfrac{\bar{z}+4}{\bar{z}-2}$, where $z \neq -4$ and $z \neq 2$.

1) Determine the coordinates of points M in the case where M and M' are confounded.
2) a- Express $|z'|$ in terms of MA and MB and verify that $\arg(z') = \arg\left(\dfrac{z-2}{z+4}\right) + 2k\pi$, ($k \in \mathbb{Z}$).

   b- Show that if M' varies on the circle (C) with center O and radius 1, then M varies on a straight line (Δ) to be determined.

   c- Determine the set of points M if z' is a strictly negative real number.

   d- Given the complex number $u = e^{-i\frac{\pi}{9}}$.
   Determine the nature of triangle MBA when u is a cubic root of z'.

II- (2.5 points)

U_1 and U_2 are two urns such that:

- U_1 contains one white ball and three black balls
- U_2 contains one red ball, three white balls, and two black balls.

One of the two urns is randomly selected:

- If the selected urn is U_1, then two balls are selected randomly and successively with replacement from U_1
- If the selected urn is U_2, then three balls are selected randomly and successively without replacement from U_2.

Consider the following events:

T: "The selected urn is U_1"

E: "Exactly two white balls are selected".

1) a- Calculate the probabilities $P(E/T)$ and $P(E \cap T)$.

   b- Show that $P\left(E \cap \bar{T}\right) = \dfrac{9}{40}$.

   c- Deduce P(E).
2) Knowing that exactly two white balls are selected, calculate the probability that they are selected from U_2.
3) Let X be the random variable equal to the number of white balls selected.

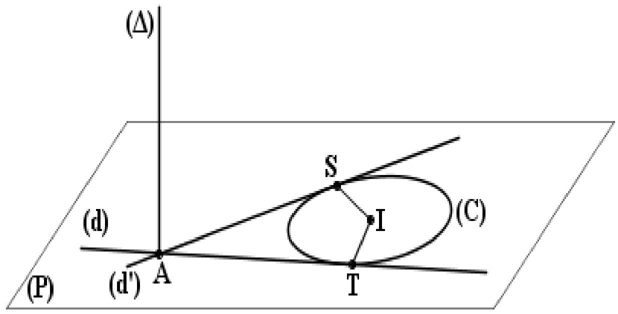
   a- Verify that $P(X=1) = \dfrac{33}{80}$.

   b- Determine $P(X \geq 1)$.

III- (2.5 points)

The space is referred to a direct orthonormal system $\left(O\,;\,\vec{i}\,,\vec{j},\vec{k}\right)$.

Consider the two lines (d) and (d') with parametric equations

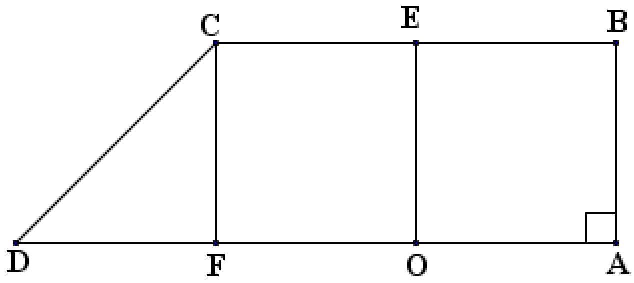


(d): $\begin{cases} x = m+1 \\ y = 2m+1 \\ z = 2m+1 \end{cases}$ and (d'): $\begin{cases} x = -t \\ y = 2t+3 \\ z = -2t-1 \end{cases}$ where m, t ∈ ℝ.

1) Show that (d) and (d') intersect at the point A(1 , 1 , 1).
2) Determine a cartesian equation of the plane (P) determined by (d) and (d').
3) Let (C) be the circle, with radius $3\sqrt{5}$, tangent to (d) at T and tangent to (d') at S.
   Let (Δ) be the line perpendicular to (P) at A.
   a- Show that the point I(1 , 10 , 1) is the center of (C).
   b- Calculate the coordinates of the two points E and F on (C) that are equidistant from (d) and (d').
   c- Show that the area of the quadrilateral ATIS is $18\sqrt{5}$.
   d- Determine the coordinates of points B on (Δ) so that the volume of the solid BATIS is 30.

IV- (3 points)

In the adjacent figure:



- ABEO and OECF are two direct squares of side 1
- $\left(\overrightarrow{AB},\overrightarrow{AO}\right) = \dfrac{\pi}{2} + 2k\pi$, where $k \in \mathbb{Z}$
- D is the symmetric of O with respect to F.

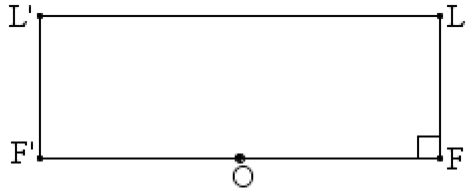
Let S be the direct plane similitude that transforms A onto C and B onto D.

1) a- Show that the ratio of S is equal to $\sqrt{2}$ and that an angle of S is $\dfrac{3\pi}{4}$.
   b- Show that O is the center of the similitude S.
   c- Determine S(E).
2) Let S^n be the transformation defined as $S^n = \underbrace{S \text{ o } S \text{ o } S \text{ o...o } S}_{n \text{ times}}$, where n is a natural number with $n \geq 2$.
   a- Determine the value of n when the image of the square OABE under S^n is a square whose area is 16 and deduce, in this case, that S^n is a negative dilation.
   b- Determine the smallest value of n so that S^n is a positive dilation.

V- (3 points)

In the adjacent figure,

- F'FLL' is a rectangle such that F'F = 4 and FL= $\sqrt{2}$
- O is the midpoint of [FF'].



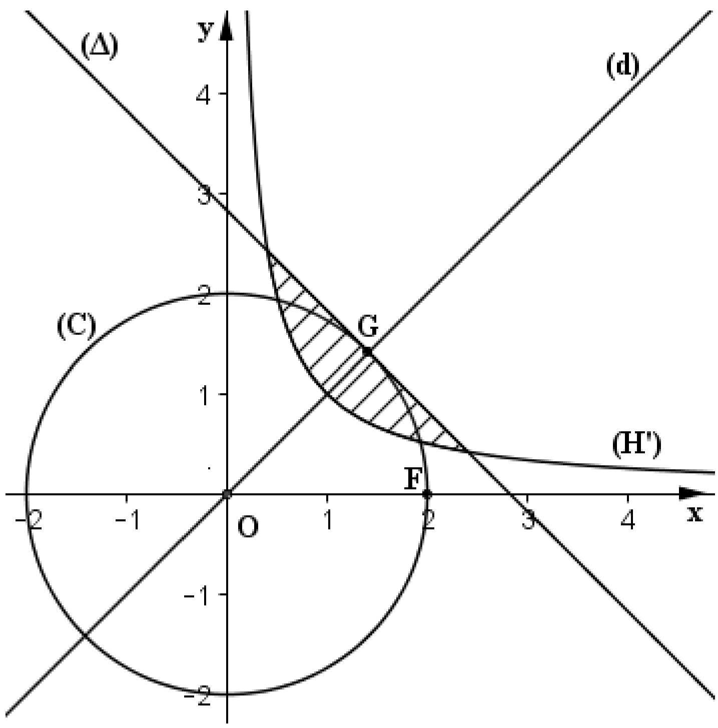
(H) is a rectangular hyperbola with foci F and F'.

1) a- Show that the point L is on (H).
   b- Show that the directrix (D) of (H) associated with the focus F is the perpendicular bisector of [OF].
2) The plane is referred to a direct orthonormal system $\left(O\,;\,\vec{i}\,,\vec{j}\right)$, such that F(2 , 0) and $L\left(2\,,\sqrt{2}\right)$.
   a- Show that $x^2 - y^2 = 2$ is an equation of (H).
   b- Determine the coordinates of the vertices of (H) and the equations of its asymptotes.
   c- Draw (H).
3) Consider the rotation R with center O and angle $\frac{\pi}{4}$.

   In the figure below:

   - (H') is the image of (H) under the rotation R
   - (d) is the first bisector
   - (C) is the circle with center O and radius 2
   - G is one of the points of intersection of (C) and (d).

   a- Show that G is the image of F under R.
   b- (Δ) is the line passing through G and perpendicular to (d).
   Determine $R^{-1}(\Delta)$, where R^{-1} is the inverse rotation of R.
   c- Calculate the volume of the solid generated by the rotation of the shaded region around the line (d).



VI- (7 points)

Consider the function f defined over]0 , + ∞[as $f(x)=\left(\frac{\ln x}{x}\right)^2$.

Denote by (C) the representative curve of f in an orthonormal system $(O;\vec{i},\vec{j})$.

Part A

Consider the differential equation (E): $xy'+2y=\frac{2\ln x}{x^2}$.

1) Show that $f'(x)=\frac{2(1-\ln x)\ln x}{x^3}$.
2) Show that $x f'(x)+2f(x)=\frac{2\ln x}{x^2}$.
3) Let $y=z+f(x)$ with $z\neq 0$.
   a- Show that a differential equation (E') satisfied by z is $\frac{z'}{z}=-\frac{2}{x}$.
   b- Solve (E') and deduce the general solution of (E).

Part B

1) Determine $\lim\limits_{\substack{x\to 0\\ x>0}} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$. Deduce the two asymptotes to (C).
2) a- Verify that f is strictly increasing over]1 , e[.
   b- Set up the table of variations of f over]0 , + ∞[and verify that $f(x)\geq 0$.
   c- Draw (C).

Part C

For all natural numbers n (n > 0), consider the sequence (I_n) defined as $I_n=\int_1^e \frac{(\ln x)^n}{x^n}dx$.

1) Calculate I_1.
2) a- Knowing that $\ln x < x$ for all $x \in [1 , e]$, prove that the sequence (I_n) is decreasing.
   b- Show, for all natural numbers n (n > 0), that $I_n \geq 0$.
   c- Deduce that the sequence (I_n) is convergent.
3) a- Knowing that $\frac{(\ln x)^n}{x^n}\leq\frac{1}{x^n}$ for all $1 \leq x \leq e$, show that $0\leq I_n\leq\frac{1-e^{-n+1}}{n-1}$.
   b- Calculate $\lim\limits_{n\to+\infty} I_n$.

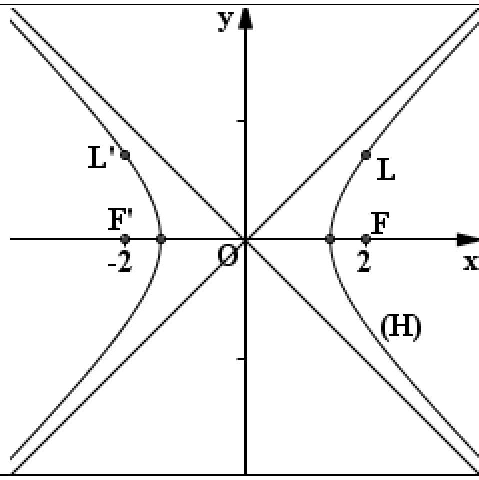
عدد المسائل: ست — أسس تصحيح مسابقة الرياضيات

I	Answers	M
1	$z=\frac{\bar{z}+4}{\bar{z}-2}$; $z\bar{z}-2z=\bar{z}+4$ Let $z=x+iy$; $x^2+y^2-2x-2iy=x-iy+4$; $x^2+y^2-3x-4-iy=0$ then $y=0$ and $x=-1$ or $x=4$; $M(-1,0)$ or $M(4,0)$	1
2a	$\lvert z'\rvert=\left\lvert\frac{\bar{z}+4}{\bar{z}-2}\right\rvert=\frac{\lvert\bar{z}+4\rvert}{\lvert\bar{z}-2\rvert}=\frac{\lvert\overline{z+4}\rvert}{\lvert\overline{z-2}\rvert}=\frac{\lvert z+4\rvert}{\lvert z-2\rvert}=\frac{AM}{BM}$. $\arg(z')=\arg\left(\frac{\bar{z}+4}{\bar{z}-2}\right)=\arg\overline{\left(\frac{z+4}{z-2}\right)}=-\arg\left(\frac{z+4}{z-2}\right)=\arg\left(\frac{z-2}{z+4}\right)+2k\pi \quad k\in\mathbb{Z}$	1
2b	$OM'=1$ then $AM=BM$ then M moves on (Δ) the perpendicular bisector of [AB].	0.5
2c	$\arg(z')=(2k+1)\pi$; $\arg\left(\frac{z-2}{z+4}\right)=(2k+1)\pi$; $\left(\overrightarrow{AM},\overrightarrow{BM}\right)=(2k+1)\pi \quad k\in\mathbb{Z}$ then M moves on the segment [AB] deprived from A and B.	0.5
2d	$z'=u^3=e^{-i\frac{\pi}{3}}$; $\lvert z'\rvert=1$ then $AM=BM$ and $\arg(z')=\frac{-\pi}{3}$ then $\left(\overrightarrow{AM},\overrightarrow{BM}\right)=-\frac{\pi}{3}$. So MBA is an equilateral triangle	1

II	Answers	M
1a	$P(E/T)=\frac{1}{4}\times\frac{1}{4}=\frac{1}{16}$; $P(E\cap T)=P(T)\times P(E/T)=\frac{1}{2}\times\frac{1}{16}=\frac{1}{32}$	1
1b	$P(E\cap\bar{T})=P(\bar{T})\times P(E/\bar{T})=\frac{1}{2}\times\left(\frac{3}{6}\times\frac{2}{5}\times\frac{3}{4}+\frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}+\frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}\right)=\frac{9}{40}$.	1
1c	$P(E)=P(E\cap T)+P(E\cap\bar{T})=\frac{1}{32}+\frac{9}{40}=\frac{41}{160}$	0.5
2	$P(\bar{T}/E)=\frac{P(\bar{T}\cap E)}{P(E)}=\frac{\frac{9}{40}}{\frac{41}{160}}=\frac{36}{41}$	0.5
3a	$P(X=1)=\frac{1}{2}\times\left(\frac{1}{4}\times\frac{3}{4}+\frac{3}{4}\times\frac{1}{4}\right)+\frac{1}{2}\times\left(\frac{3}{6}\times\frac{2}{5}\times\frac{3}{4}+\frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}+\frac{3}{6}\times\frac{3}{5}\times\frac{2}{4}\right)=\frac{33}{80}$	1
3b	$P(X\geq 1)=P(X=1)+P(X=2)+P(X=3)=\frac{33}{80}+\frac{41}{160}+\frac{1}{2}\times\left(\frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}\right)=\frac{111}{160}$ OR : $P(X\geq 1)=1-P(X=0)=1-\left(\frac{1}{2}\left(\frac{3}{4}\times\frac{3}{4}\right)+\frac{1}{2}\left(\frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}\right)\right)=\frac{111}{160}$	1

III	Answers	M
1	$\vec{V}_d(1,2,2)$ and $\vec{V}_{d'}(-1,2,-2)$ are non collinear vectors. For $m=0,\ A\in(d)$. For $t=-1\ A\in(d)$	0.5
2	Let $M(x,y,z)\in(P)$ then $\overrightarrow{AM}.\left(\vec{V}\wedge\vec{V'}\right)=0 \Leftrightarrow \begin{vmatrix} x-1 & y-1 & z-1 \\ 1 & 2 & 2 \\ -1 & 2 & -2 \end{vmatrix}=0$; $(P): -2x+z+1=0$	1
3a	$-2x_I+z_I+1=0$ then $I\in(P)$ $d(I,(d))=\frac{\left\|\overrightarrow{IA}\wedge\vec{V}_d\right\|}{\left\|\vec{V}_d\right\|}=3\sqrt{5}=R$; $d(I,(d'))=\frac{\left\|\overrightarrow{IA}\wedge\vec{V}_d\right\|}{\left\|\vec{V}_d\right\|}=3\sqrt{5}=R$	1
3b	$(AI):\begin{cases} x=1 \\ y=n+1 \\ z=1 \end{cases}$; $E(1,n+1,1)$; $IE=3\sqrt{5}$; $(n-9)^2=45$; $n=9+3\sqrt{5}$ or $n=9-3\sqrt{5}$ $E(1,10+3\sqrt{5},1)$ and $F(1,10-3\sqrt{5},1)$	1
3c	$A_{ATIS}=2\times A_{ATI}=AT\times IT=\sqrt{IA^2-IT^2}\times 3\sqrt{5}=\sqrt{81-45}\times 3\sqrt{5}=18\sqrt{5}$ units of area	0.5
3d	$(\Delta):\begin{cases} x=-2k+1 \\ y=1 \\ z=k+1 \end{cases}$ $(k\in\mathbb{R})$; $B(-2k+1,1,k+1)$; $\overrightarrow{AB}(-2k,0,k)$; $V_{BATIS}=\frac{1}{3}\times A_{ATIS}\times AB$; $6\sqrt{5}AB=30$; $6\sqrt{5}\sqrt{5k^2}=30$; $30\lvert k\rvert=30$; $k=1$ or $k=-1$ then $B(-1,1,2)$ or $B(3,1,0)$ or $V=2.V'=2.\frac{1}{6}\left\|\overrightarrow{BA}.\left(\overrightarrow{AS}\wedge\overrightarrow{AI}\right)\right\|=30...$	1

IV	Answers	M
1a	$k=\frac{CD}{AB}=\sqrt{2}$; $\alpha=\left(\overrightarrow{AB},\overrightarrow{CD}\right)+2k\pi=\left(\overrightarrow{FC},\overrightarrow{CD}\right)+2k\pi=\pi+\left(\overrightarrow{CF},\overrightarrow{CD}\right)+2k\pi=\frac{-\pi}{4}+\pi=\frac{3\pi}{4}+2k\pi$	1
1b	$\frac{OC}{OA}=\sqrt{2}$ and $\left(\overrightarrow{OA},\overrightarrow{OC}\right)=\frac{\pi}{2}+\frac{\pi}{4}+2k\pi=\frac{3\pi}{4}+2k\pi$ then $S(O)=O$. So O is the center of S.	1
1c	S(E) = E' ; OABE is a direct square then OCDE' is a direct square with center F then E' is the symmetric of C w.r.t. F.	1
2a	Area of image $=\left(k^2\right)^n\times$ Area of OABE ; $16=k^{2n}$; $16=2^n$; $n=4$. $S^4=S\left(O;\left(\sqrt{4}\right)^4;4\times\frac{3\pi}{4}\right)=H(O;-4)$ is a negative dilation.	1.5
2b	$S^n=S\left(O;\left(\sqrt{2}\right)^n;\frac{3n\pi}{4}\right)$ is a positive dilation if $\frac{3n\pi}{4}=2k\pi$; $3n=8k$; n is a multiple of 8; $n\in\{8;16;.....\}$; the smallest value for n is 8	1.5

V	Answers	M
1a	$FF' = 4; c = 2;\ a = b = \sqrt{2}$; $LF' - LF = 3\sqrt{2} - \sqrt{2} = 2\sqrt{2} = 2a$ then $L \in (H)$.	1
1b	$\overrightarrow{OF} = e\overrightarrow{OS}$ and $\overrightarrow{OS} = e\overrightarrow{OK}$ then $\overrightarrow{OF} = e^2\overrightarrow{OK} = 2\overrightarrow{OK}$ where K is the intersection point of the directrix with (FF') and S is a vertex of (H). OR $d(O,(D)) = 1 = \frac{a^2}{c}$ and $(D) \perp (FF')$: focal axis	0.5
2a	$O(0,0)$ is the center of (H); $(x'Ox)$ is the focal axis ; $a = b = \sqrt{2}$ then $(H): x^2 - y^2 = 2$	0.5
2b	Vertices $A(\sqrt{2},0)$ and $A'(-\sqrt{2},0)$ Asymptotes $y = x$ and $y = -x$	1
2c		0.5
3a	$F = (C) \cap (x'Ox)$; $R(F) = R(C) \cap R(x'Ox) = (C) \cap (d) = G$ such that $\left(\overrightarrow{OF}, \overrightarrow{OG}\right) = \frac{\pi}{4}$	0.5
3b	(Δ) passes through G and perpendicular to (d) $R^{-1}(\Delta)$ passes through $R^{-1}(G)$ and perpendicular to $R^{-1}(d)$ $R^{-1}(\Delta)$ passes through F and perpendicular to $(x'Ox)$	1
3c	$V = \pi \int_{\sqrt{2}}^{2} y^2 dx = \frac{4\sqrt{2} - 4}{3}\pi$ units of volume	1

VI	Answers	M
A1	$f'(x) = \frac{2x\ln x - 2x(\ln x)^2}{x^4} = \frac{2\ln x - 2(\ln x)^2}{x^3} = \frac{2(1-\ln x)\ln x}{x^3}$	1
A2	$x f'(x) + 2f(x) = x\left(\frac{2\ln x - 2(\ln x)^2}{x^3}\right) + 2\left(\frac{\ln x}{x}\right)^2 = \frac{2\ln(x)}{x^2}$	1
A3a	$y = z + f(x)$; $y' = z' + f'(x)$; $xy' + 2y = \frac{2\ln x}{x^2}$; $xz' + xf'(x) + 2z + 2f(x) = \frac{2\ln x}{x^2}$; $xz' + 2z = 0$ then $\frac{z'}{z} = \frac{-2}{x}$	1

<table>
<tr><td>A3b</td><td>$\int \frac{z'}{z}dx=\int \frac{-2}{x}dx$; $\ln|z|=-2\ln|x|+K_1=-\ln x^2+\ln e^{K_1}=\ln\frac{K_2}{x^2}$; $z=\frac{C}{x^2}$; $y=\frac{C}{x^2}+\frac{2\ln x}{x^2}$</td><td>1</td></tr>
<tr><td>B1</td><td>$\lim\limits_{\substack{x\to 0\\ x>0}} f(x)=+\infty$; $\lim\limits_{x\to+\infty} f(x)=0$; $x=0$ and $y=0$ are the asymptotes to (C).</td><td>1</td></tr>
<tr><td>B2a</td><td>$f'(x)=\frac{2(1-\ln x)\ln x}{x^3}$
<table>
<tr><td>x</td><td>0</td><td></td><td>1</td><td></td><td>e</td><td></td><td>+∞</td></tr>
<tr><td>1 − lnx</td><td>||</td><td>+</td><td></td><td>+</td><td>0</td><td>−</td><td></td></tr>
<tr><td>lnx</td><td>||</td><td>−</td><td>0</td><td>+</td><td></td><td>+</td><td></td></tr>
<tr><td>f '(x)</td><td>||</td><td>−</td><td>0</td><td>+</td><td>0</td><td>−</td><td></td></tr>
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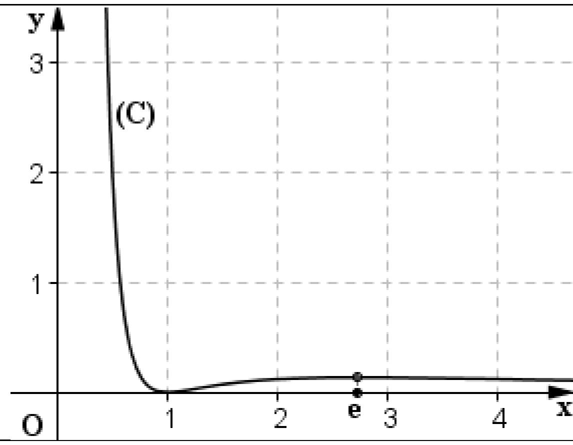
f is strictly increasing over]1 , e[.</td><td>1</td></tr>
<tr><td>B2b</td><td>
<table>
<tr><td>x</td><td>0</td><td></td><td>1</td><td></td><td>e</td><td></td><td>+∞</td></tr>
<tr><td>f '(x)</td><td>||</td><td>−</td><td>0</td><td>+</td><td>0</td><td>−</td><td></td></tr>
<tr><td>f(x)</td><td>||</td><td>+∞ ↘</td><td>0</td><td>↗</td><td>$\frac{1}{e^2}$</td><td>↘</td><td>0</td></tr>
</table>
Minimum of $f(x)$ over $]0;+\infty[$ is 0. Then $f(x)\ge 0$ for $x>0$.</td><td>1.5</td></tr>
<tr><td>B2c</td><td>
</td><td>1.5</td></tr>
<tr><td>C1</td><td>$I_1=\int_1^e \frac{\ln x}{x}dx=\left[\frac{(\ln x)^2}{2}\right]_1^e=\frac{1}{2}.$</td><td>0.5</td></tr>
<tr><td>C2a</td><td>$I_{n+1}-I_n=\int_1^e \frac{(\ln x)^{n+1}}{x^{n+1}}-\frac{(\ln x)^n}{x^n}dx=\int_1^e \frac{(\ln x)^n}{x^n}\left(\frac{\ln x-x}{x}\right)dx\le 0$
since $(\ln x)^n\ge 0; x^n>0$ and $\ln x-x<0$ for $x\in[1,e]$. Then the sequence (In) is decreasing</td><td>1</td></tr>
<tr><td>C2b</td><td>For $x\in[1,e], (\ln x)^n\ge 0$ and $x^n>0$ then $I_n=\int_1^e \frac{(\ln x)^n}{x^n}dx\ge 0.$</td><td>1</td></tr>
<tr><td>C2c</td><td>The sequence (I_n) is decreasing and bounded below by zero then it is convergent.</td><td>0.5</td></tr>
<tr><td>C3a</td><td>$\frac{(\ln x)^n}{x^n}\le\frac{1}{x^n}$; $\int_1^e \frac{(\ln x)^n}{x^n}dx\le\int_1^e \frac{1}{x^n}dx$; $I_n\le\left[\frac{x^{-n+1}}{-n+1}\right]_1^e=\frac{1-e^{-n+1}}{n-1}$</td><td>1</td></tr>
<tr><td>C3b</td><td>$0\le I_n\le\frac{1-e^{-n+1}}{n-1}$ then $\lim\limits_{n\to+\infty} I_n=0$</td><td>1</td></tr>
</table>