

الاسم:
الرقم:

مسابقة في مادة الرياضيات
المدة: أربع ساعات

عدد المسائل: ست

ملاحظة: - يسمح باستعمال آلة حاسبة غير قابلة للبرمجة او اختزان المعلومات او رسم البيانات.
- يستطيع المرشح الإجابة بالترتيب الذي يناسبه (دون الالتزام بترتيب المسائل الواردة في المسابقة).

I- (2 points)

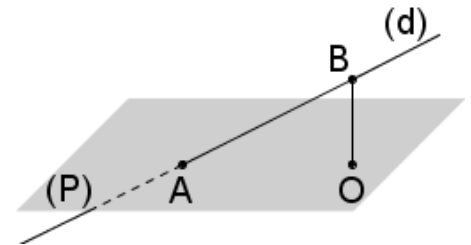
In the table below, only one of the proposed answers to each question is correct. Write down the number of each question and give, **with justification**, the answer corresponding to it.

N°	Questions	Answers		
		a	b	c
1	z is a complex number. One of the roots of the equation $z^4 + z^2 - i\sqrt{3} = 0$ is	$e^{i\frac{\pi}{6}}$	i	$e^{-i\frac{\pi}{6}}$
2	Denote by f the function defined on \mathbb{R} as $f(x) = \frac{1}{x^2 + 4x + 8}$. An antiderivative of f is	$\frac{1}{2} \arctan\left(\frac{x+2}{2}\right)$	$\frac{1}{4} \arctan\left(\frac{x+2}{2}\right)$	$\arctan(x+2)$
3	m is a real number ($m > 1$). If $J = \int_m^{m+1} \frac{1}{x} dx$, then J belongs to the interval	$[m, m+1]$	$\left]0, \frac{1}{m+1}\right[$	$\left[\frac{1}{m+1}, \frac{1}{m}\right]$
4	z is a complex number. If $z = 1 + \cos \theta + i \sin \theta$, with $\pi < \theta < 2\pi$, then $ z =$	$2 \cos\left(\frac{\theta}{2}\right)$	$-2 \cos\left(\frac{\theta}{2}\right)$	$\sqrt{2}$

II- (2 points)

In the space referred to an orthonormal system $(O; \vec{i}, \vec{j}, \vec{k})$, consider the plane (P) with equation $x + z = 0$

and the line (d) with parametric equations $\begin{cases} x = -t + 1 \\ y = -2t \\ z = 3t + 1 \end{cases} (t \in \mathbb{R})$.



1) Calculate the coordinates of point A, the intersection of (d) and (P).

2) The point B(1, 0, 1) is on (d).

a- Show that O is the orthogonal projection of B on (P).

b- Deduce a system of parametric equations of the line (Δ), the orthogonal projection of (d) on (P).

3) The point J(-5, 2, 5) is on (P).

Calculate the volume of the tetrahedron OBJA.

4) In the plane (P), consider the hyperbola (H) with foci O and A and eccentricity $e = 3$.

a- Verify that the point I(1, 1, -1) is the center of (H).

b- Calculate the coordinates of S and G, the two vertices of (H).

III- (3 points)

Consider two fair cubic dice. The faces of each die are numbered from 1 to 6.

The two dice are rolled.

Denote by X the random variable that is defined as follows:

- if the two numbers shown on the dice are different, then X is equal to the greater between them;
- if the two numbers shown on the dice are equal, then X is equal to one them.

For example,

- if the two numbers shown on the dice are 2 and 3, then $X = 3$
- if the two numbers shown on the dice are 4 and 4, then $X = 4$

1) a- Calculate the probabilities $P(X = 1)$ and $P(X = 2)$.

b- Prove that $P(X \leq 3) = \frac{1}{4}$.

2) In this part, consider an urn U that contains 6 balls: 4 red and 2 blue.

The two dice are rolled:

- if $X \leq 3$, then 3 balls are randomly and simultaneously selected from U ;
- if $X > 3$, then 3 balls are selected randomly and successively with replacement from U .

Consider the following events:

A: " $X \leq 3$ "

S: "The 3 selected balls have the same color"

a- Calculate $P\left(\frac{S}{A}\right)$ and $P(A \cap S)$.

b- Verify that $P(\bar{A} \cap S) = \frac{1}{4}$ and calculate $P(S)$.

c- Knowing that $X > 3$, calculate the probability that the three selected balls do not have the same color.

IV- (3 points)

In the adjacent figure,

- F and F' are two fixed points so that $FF' = 2$.
- N is a variable point on the circle with center F' and radius 4.
- The perpendicular bisector of $[NF]$ intersects $[F'N]$ at M .
- B is a fixed point so that $BF'F$ is an equilateral triangle.

1) a- Show that $MF + MF' = 4$.

b- Deduce that M moves on a conic (E) whose nature, foci and center O are to be determined.

2) Let A be the symmetric of O with respect to F .

a- Show that A is one of the vertices of (E).

b- Determine the non-focal axis of (E) and verify that B is one of the vertices of (E).

c- Draw (E).

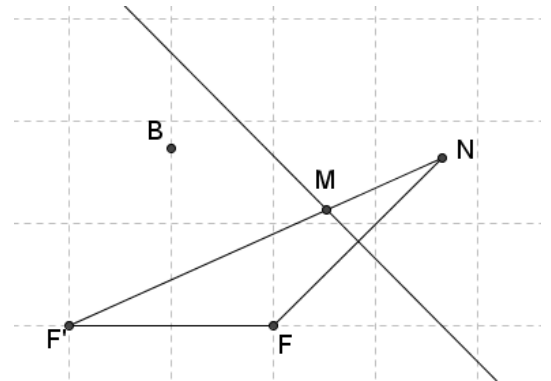
3) The plane is referred to a direct orthonormal system $(O; \vec{i}, \vec{j})$ with $\vec{i} = \overrightarrow{OF}$.

a- Verify that $\frac{x^2}{4} + \frac{y^2}{3} = 1$ is an equation of (E).

b- Write an equation of (d) the directrix of (E) associated to F .

c- Let $L(\alpha, \beta)$ be a point on (E) where α and β are two real numbers with $\beta \neq 0$.

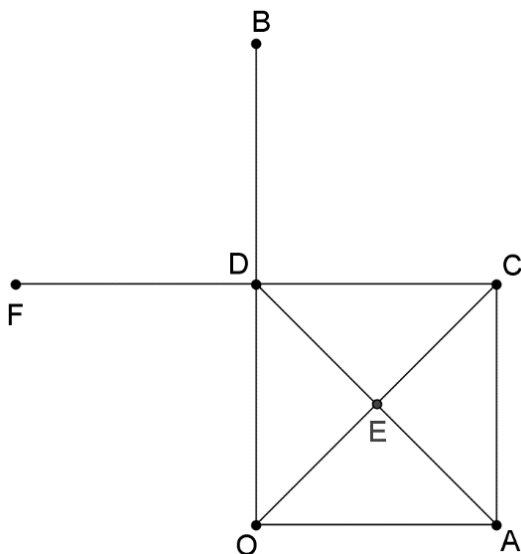
Write an equation of the tangent (T) to (E) at L .



V- (3 points)

In the figure below,

- OACD is a direct square with center E and side 2.
- F is the symmetric of C with respect to D.
- B is the symmetric of O with respect to D.



Denote by S the direct plane similitude of center O that maps A onto B.

Part A

- 1) a- Calculate the ratio k and an angle α of S.
 b- Verify that $S(E) = F$.
 c- Show that the triangle OBF is right isosceles.
- 2) Consider the direct plane similitude $S' \left(E, 2, \frac{\pi}{2} \right)$ and the transformation $h = S \circ S'$.

Denote by W the center of h. Show that $\overline{WF} = -4\overline{WE}$.

Part B

The plane is referred to a direct orthonormal system $(O; \vec{u}, \vec{v})$ with $\vec{u} = \frac{1}{2}\overline{OA}$.

- 1) Show that the complex form of h is $z' = -4z + 2 + 6i$ and deduce the affix of W.
- 2) For all $n \in \mathbb{N}$, consider the numerical sequence (d_n) defined as $d_n = WE_n$ where $E_0 = E$ and $E_{n+1} = h(E_n)$.

a- Verify that $d_0 = \frac{\sqrt{10}}{5}$.

- b- Show that (d_n) is a geometric sequence of common ratio 4.
 c- Determine the number of points E_n such that $d_n < 2019$.

VI- (7 points)

Part A

Consider the differential equation (E): $y'' - 2y' + y = e^{2x}$ and let $y = z + e^{2x}$.

- 1) Write a differential equation (E') satisfied by z .
- 2) Determine the general solution of (E).
- 3) Determine the particular solution of (E) whose representative curve, in an orthonormal system, has at the point $A(0, -2)$ a tangent parallel to the x -axis.

Part B

Let f be the function defined on \mathbb{R} as $f(x) = e^{2x} + (x-3)e^x$.

Denote by (C) the representative curve of f in an orthonormal system $(O; \vec{i}, \vec{j})$.

- 1) Determine $\lim_{x \rightarrow +\infty} f(x)$ and $\lim_{x \rightarrow +\infty} \frac{f(x)}{x}$.
- 2) Determine $\lim_{x \rightarrow -\infty} f(x)$ and deduce an asymptote to (C).
- 3) Let g be the function defined on \mathbb{R} as $g(x) = x - 2 + 2e^x$.
 - a- Set up the table of variations of the function g .
 - b- Calculate $g(0)$ then deduce, according to the values of x , the sign of $g(x)$.
- 4) Verify that $f'(x) = e^x g(x)$ and set up the table of variations of the function f .
- 5) Show that the equation $f(x) = 0$ has, on \mathbb{R} , a unique root α . Verify that $0.7 < \alpha < 0.8$.
- 6) Draw the curve (C).
 - a- Prove that f has, over $[0, +\infty[$, an inverse function f^{-1} and determine its domain of definition.
 - b- Draw the representative curve (C') of f^{-1} in the same system $(O; \vec{i}, \vec{j})$.
 - c- Calculate, in terms of α , the area of the region bounded by (C'), the x -axis and the y -axis.

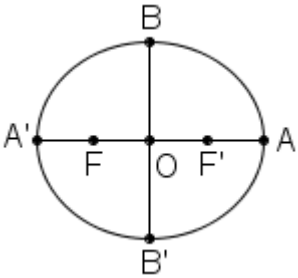
Part C

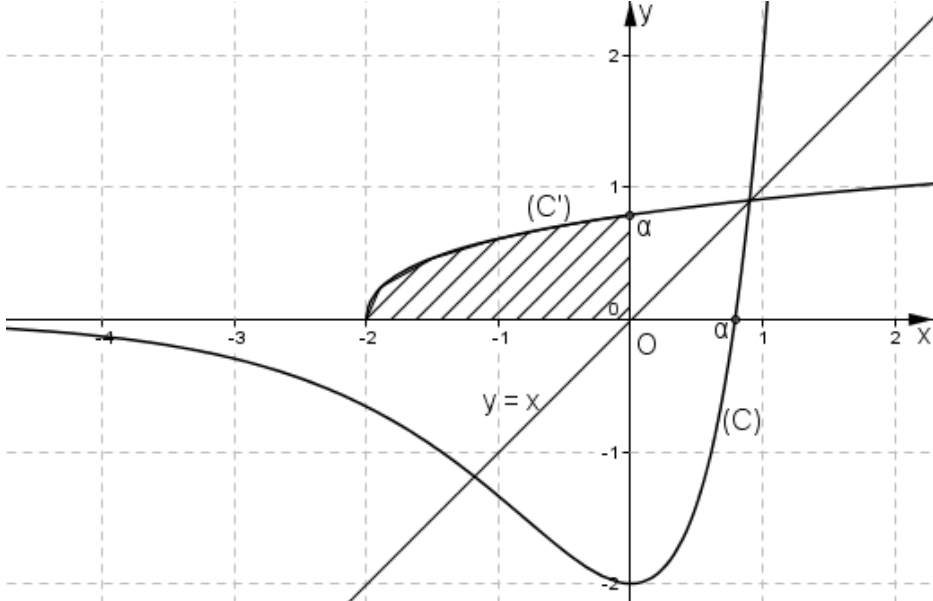
Let h be the function given by $h(x) = \arcsin(f^{-1}(x))$.

Show that the domain of definition of the function h is $[-2, e^2 - 2e]$.

أسس تصحيح مسابقة الرياضيات

Q.I	Answers	4pts
1	$z = e^{i\frac{\pi}{6}}$. Then a	1
2	$\int \frac{dx}{x^2+4x+8} = \int \frac{1}{2^2+(x+2)^2} dx = \frac{1}{4} \int \frac{1}{1+(\frac{x+2}{2})^2} = \frac{1}{2} \arctan(\frac{x+2}{2}) + C$. Then a OR find the derivative of $\frac{1}{2} \arctan(\frac{x+2}{2})$	1
3	$m \leq x \leq m+1, \frac{1}{m+1} \leq \frac{1}{x} \leq \frac{1}{m}$ then $\frac{1}{m+1} \int_m^{m+1} dx \leq \int_m^{m+1} \frac{1}{x} dx \leq \frac{1}{m} \int_m^{m+1} dx$ so $\frac{1}{m+1} \leq J \leq \frac{1}{m}$. Then c	1
4	$z = 2\cos^2(\frac{\theta}{2}) + 2\sin(\frac{\theta}{2}) \cdot \cos(\frac{\theta}{2}) \cdot i = -2\cos(\frac{\theta}{2})[-\cos(\frac{\theta}{2}) - i \cdot \sin(\frac{\theta}{2})] = -2\cos(\frac{\theta}{2}) e^{i(\pi+\frac{\theta}{2})}$ $(\pi < \theta < 2\pi \text{ donc } \frac{\pi}{2} < \frac{\theta}{2} < \pi; \cos(\frac{\theta}{2}) < 0)$ alors $ z = -2\cos(\frac{\theta}{2})$. Then b OR $ z = \sqrt{(1+\cos\theta)^2 + (\sin\theta)^2} = \sqrt{2+2\cos\theta} = \sqrt{4(\cos\frac{\theta}{2})^2} = 2\cos(\frac{\theta}{2}) $ $= -2\cos(\frac{\theta}{2}) \text{ car } \frac{\pi}{2} < \frac{\theta}{2} < \pi$. OR take a particular value satisfied for one solution only.	1
Q.II	Answers	4pts
1	$A \in (d)$ then $A(-t+1; -2t; 3t+1)$ and $A \in (P), x_A + z_A = 0, t = -1$ so $A(2; 2; -2)$.	0.5
2a	O is a point on (P) and $\overrightarrow{OB}(1; 0; 1) = \overrightarrow{N_P}$	0.5
2b	$(\Delta) \equiv (OA): \begin{cases} x = 2m \\ y = 2m \\ z = -2m \end{cases} (m \in \mathbb{R})$ OR $\begin{cases} x = 2m+2 \\ y = 2m+2 \\ z = -2m-2 \end{cases} (m \in \mathbb{R})$ OR $\begin{cases} x = m \\ y = m \\ z = -m \end{cases} (m \in \mathbb{R})$	0.5
3	$V_{O_BJA} = \frac{1}{6} \det(\overrightarrow{OB}, \overrightarrow{OA}, \overrightarrow{OJ}) = \frac{28}{6} = \frac{14}{3}$ volume units.	1
4a	The center of (H) is the midpoint of [OA] then $I(1; 1; -1)$.	0.5
4b	$S \in (OA)$ then $S(m; m; -m)$. $e = \frac{c}{a} = 3$ and $c = OI = \sqrt{3}$ so $a = \frac{1}{\sqrt{3}}$ $IS = a$ thus $3(m-1)^2 = \frac{1}{3}; m = \frac{4}{3}$ or $m = \frac{2}{3}$. Hence, $S(\frac{4}{3}; \frac{4}{3}; -\frac{4}{3})$ and $G(\frac{2}{3}; \frac{2}{3}; -\frac{2}{3})$	1
Q.III	Answers	6pts
1a	$P(X=1) = \frac{1}{36}$; $P(X=2) = \frac{3}{36} = \frac{1}{12}$	1
1b	$P(X \leq 3) = P(X=1) + P(X=2) + P(X=3) = \frac{9}{36} = \frac{1}{4}$	1
2a	$P(S/A) = \frac{C_4^3}{C_6^3} = \frac{1}{5}$ $P(A \cap S) = P(A) \times P(S/A) = \frac{1}{4} \times \frac{1}{5} = \frac{1}{20}$	1
2b	$P(\bar{A} \cap S) = P(\bar{A}) \times P(S/\bar{A}) = \frac{3}{4} \times (\frac{4^3}{6^3} + \frac{2^3}{6^3}) = \frac{1}{4}$ $P(S) = P(\bar{A} \cap S) + P(A \cap S) = \frac{1}{4} + \frac{1}{20} = \frac{3}{10}$	1.5
2c	$P(\bar{S}/\bar{A}) = 1 - P(S/\bar{A}) = 1 - \frac{P(S \cap \bar{A})}{P(\bar{A})} = 1 - \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{2}{3}$ OR $P(\bar{S}/\bar{A}) = \frac{P(\bar{S} \cap \bar{A})}{P(\bar{A})} = \frac{\frac{3}{4} \times \frac{2}{3}}{1 - \frac{1}{4}} = \frac{2}{3}$	1.5

Q.IV	Answers	6pts
1a	$MF + MF' = MN + MF' = 4$	0.5
1b	$MF + MF' = 4 = 2a$ which is a constant greater than $FF' = 2$. So, M belongs an ellipse (E) with foci F and F'. The center O is the midpoint of $[FF']$.	1
2a	$AF + AF' = 1 + 3 = 4$ then $A \in (E)$ and A is a point on the focal axis (FF'). Then A is a principal vertex of (E).	0.5
2b	The non-focal axis is the line (OB) passing through O and perpendicular to the focal axis (FF') $a = \frac{4}{2} = 2, c = \frac{FF'}{2} = 1$ et $b = \sqrt{a^2 - c^2} = \sqrt{3}$ B is a point of the non-focal axis of (E). OBF is a semi-equilateral triangle and $OB = \frac{BF \times \sqrt{3}}{2} = \frac{2 \times \sqrt{3}}{2} = \sqrt{3}$ Thus, B is a vertex of (E). OR $BF + BF' = 4$ and B is a point of the non-focal axis of (E).	1
2c	The second vertex on the non-focal axis of (E) other than B is the symmetric of B with respect to O. The second vertex on the focal axis of (E) other than A is the symmetric of A with respect to O.	1
		
3a	(E) is an ellipse of center $O(0; 0)$ and focal axis (FF') $\equiv (x'x)$ Then (E) : $\frac{(x-0)^2}{2^2} + \frac{(y-0)^2}{\sqrt{3}^2} = 1$ So $\frac{x^2}{4} + \frac{y^2}{3} = 1$	1
3b	(d) : $x = \frac{a^2}{c}$ then (d) : $x = 4$	0.5
3c	(T) : $\frac{\alpha x}{4} + \frac{\beta y}{3} = 1 ; y = \frac{-3\alpha}{4\beta}x + \frac{3}{\beta}$	0.5
Q.V	Answers	6pts
A1a	The ratio $k = \frac{OB}{OA} = 2$ The angle $\alpha = (\overrightarrow{OA}; \overrightarrow{OB})(2\pi) = \frac{\pi}{2}(2\pi)$	1
A1b	$\frac{OF}{OE} = 2 = k$ $(\overrightarrow{OE}; \overrightarrow{OF}) = \frac{\pi}{4} + \frac{\pi}{4}(2\pi) = \frac{\pi}{2}(2\pi) = \alpha$ Then $S(E) = F$.	0.5
A1c	$S(OAE) = OBF$ and OAE is right isosceles triangle. Then OBF is isosceles right triangle.	0.5
A2	$h = \text{sim}(W, 4, \pi) = \text{hom}(W, -4)$. $h(E) = S(S'(E)) = S(E) = F$ then $\overrightarrow{WF} = -4\overrightarrow{WE}$	1
B1	$z' = -4z + 2 + 6i$ $Z_W = \frac{2}{5} + \frac{6}{5}i$	1
B2a	$d_0 = WE = \frac{\sqrt{10}}{5}$	0.5
B2b	$d_{n+1} = 4d_n$ then (d_n) is a geometric sequence of common ratio 4.	0.5
B2c	$d_n = d_0 \times 4^n = \frac{\sqrt{10}}{5} \times 4^n$ $d_n < 2019 ; \frac{\sqrt{10}}{5} \times 4^n < 2019$ then $n < 5.8$ So $n = 5$ then the number of points is 6.	1

Q.VI	Answers	14pts												
A1	$z'' - 2z' + z = 0$	0.5												
A2	$r^2 - 2r + 1 = 0, r = 1, z = (Ax + B)e^x, y = z + e^{2x} = (Ax + B)e^x + e^{2x}$	1												
A3	$y(0) = -2, B = -3;$ $y' = Ae^x + (Ax + B)e^x + 2e^{2x}, y'(0) = 0, A = 1.$ Then $y = (x - 3)e^x + e^{2x}$	1												
B1	$\lim_{x \rightarrow +\infty} f(x) = +\infty; \lim_{x \rightarrow +\infty} \frac{f(x)}{x} = +\infty$	1												
B2	$\lim_{x \rightarrow -\infty} f(x) = 0, y = 0$ HA	1												
B3a	$g'(x) = 1 + 2e^x > 0$ <table border="1" style="margin-left: 20px;"> <tr> <td>x</td> <td>$-\infty$</td> <td>0</td> <td>$+\infty$</td> </tr> <tr> <td>$g'(x)$</td> <td></td> <td>+</td> <td></td> </tr> <tr> <td>$g(x)$</td> <td>$-\infty$</td> <td>0</td> <td>$+\infty$</td> </tr> </table>	x	$-\infty$	0	$+\infty$	$g'(x)$		+		$g(x)$	$-\infty$	0	$+\infty$	1
x	$-\infty$	0	$+\infty$											
$g'(x)$		+												
$g(x)$	$-\infty$	0	$+\infty$											
B3b	$g(0) = 0.$ $g(x) < 0$ for $x < 0, g(x) = 0$ for $x = 0, g(x) > 0$ for $x > 0.$	1												
B4	$f'(x) = 2e^{2x} + e^x + (x - 3)e^x = e^x g(x).$ <table border="1" style="margin-left: 20px;"> <tr> <td>x</td> <td>$-\infty$</td> <td>0</td> <td>$+\infty$</td> </tr> <tr> <td>$f'(x)$</td> <td></td> <td>-</td> <td>+</td> </tr> <tr> <td>$f(x)$</td> <td>0</td> <td>-2</td> <td>$+\infty$</td> </tr> </table>	x	$-\infty$	0	$+\infty$	$f'(x)$		-	+	$f(x)$	0	-2	$+\infty$	1.5
x	$-\infty$	0	$+\infty$											
$f'(x)$		-	+											
$f(x)$	0	-2	$+\infty$											
B5	Over $]-\infty; 0[$: $f(x) < 0$ then $f(x) = 0$ has no roots. Over $[0; +\infty[$: f is continuous and strictly increasing from $-2 < 0$ to $+\infty$ then the equation $f(x) = 0$ has a unique root α . But $f(0.7).f(0.8) = (-0.57)(0.05) < 0$ then $0.7 < \alpha < 0.8$	1.5												
B6		1.5												
B7a	Over $[0; +\infty[$: f is continue et strictly increasing then f has an inverse function f^{-1} . $D_{f^{-1}} = [-2; +\infty[$	0.5												
B7b	(C) and (C') are symmetric with respect to $(y = x)$. Figure	1												
B7c	Area = $\int_0^\alpha -f(x)dx = -\left[\frac{1}{2}e^{2x} + (x - 3)e^x - e^x\right]_0^\alpha = \left[(4 - \alpha)e^\alpha - \frac{1}{2}e^{2\alpha}\right] - \left[\frac{7}{2}\right]$ unit of area. Remark: $f(\alpha) = 0, e^\alpha = 3 - \alpha$. Then Area = $\frac{1}{2}\alpha^2 - 4\alpha + 4$ unit of area.	1												
C	$-1 \leq f^{-1}(x) \leq 1$ and $f^{-1}(x) \geq 0$ then $0 \leq f^{-1}(x) \leq 1, f(0) < x < f(1)$ since f is increasing, then $-2 \leq x \leq e^2 - 2e$.	0.5												