المركز التربوي للبحوث والإنماء	الهيئة الأكاديميّة المشتركة قسم: الرياضيات	المادة: رياضيات – لغة انكليزية الشهادة: الثانوية العامة الفرع: علوم عامة نموذج رقم: 2 / 2019 المدّة: اربع ساعات.

ملاحظة: يُسمح باستعمال آلة حاسبة غير قابلة للبرمجة أو اختزان المعلومات أو رسم البيانات.
يستطيع المرشح الإجابة بالترتيب الذي يناسبه (دون الالتزام بترتيب المسائل الوارد في المسابقة).

I- (1 point)

In the table below, only one answer to each question is correct. Write down the number of the question and so its corresponding answer. Justify your choice.

	Question	Answer a	Answer b	Answer c
1	$\arcsin\left(\sin\left(\frac{25\pi}{4}\right)\right) =$	$\frac{25\pi}{4}$	$\frac{\pi}{4}$	$\frac{-\pi}{4}$
2	$\int \frac{1}{\sqrt{-x^2-4x-3}}\,dx$	$\arcsin(x+2)+c$	$\arcsin(x-2)+c$	$\frac{1}{2}\arcsin(x-2)+c$
3	Z_1 and Z_2 are the roots of the equation: $Z^2-2(1+i)Z+k=0$, where k is a real number, then $\lvert Z_1+Z_2\rvert =$	9	$\sqrt{2}$	$2\sqrt{2}$
4	If the real and the imaginary parts of a nonzero complex number Z are of unlike signs, then an $\arg\left(\frac{Z+\bar{Z}}{Z-\bar{Z}}\right) =$	$\frac{\pi}{2}$	$-\frac{\pi}{2}$	π

II- (2 points)

In the space referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$, consider the two lines (d_1) and (d_2) with parametric equations: $(d_1)\begin{cases}x=-t+1\\y=5t+1\\z=4t-2\end{cases}$ and $(d_2)\begin{cases}x=4m-3\\y=m\\z=5m-7\end{cases}$,

where t and m are two real numbers.

1) a) Verify that (d_1) and (d_2) intersect at $A(1,1,-2)$.
   b) Prove that $x+y-z-4=0$ is an equation of the plane (P) determined by (d_1) and (d_2).

2) Let (Q) be the plane with equation $2x-y+z-2=0$.
   a) Show that the two planes (P) and (Q) are perpendicular.
   b) Let (D) be the intersection line of (P) and (Q). Determine the coordinates of H, meeting point of (D) and (d_1).
   c) Determine a point G on (d_2) so that $AG=AH$.
   d) Deduce the coordinates of a direction vector for a bisector of angle $\widehat{GAH}$.

III- (3,5 points)

Consider a fair die numbered 1 through 6, and 3 urns U_1, U_2, and U_3 containing 5 balls each. U_1 contains three black balls, U_2 contains 2 black balls and U_3 contains 1 black ball. The other balls in each urn are white.

Consider the following game: the player rolls the die.

- If the die shows the number 1, then the player selects randomly and simultaneously two balls from U_1
- If the die shows the number 3 or 6, then the player selects randomly two balls from U_2, one after another with replacement
- If the die shows 2 or 4 or 5, then the player selects randomly from U_3, two balls one after another without replacement.

Consider the following events:

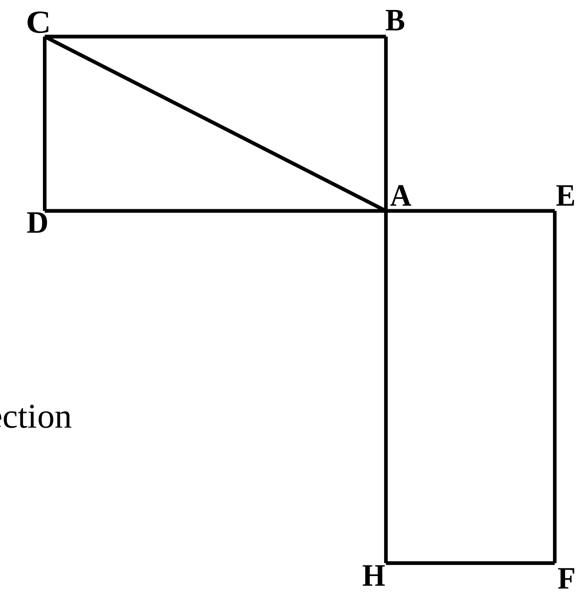
N: The selected balls have different colors.

E_i: The selected urn is U_i (i = 1, 2, or 3).

1) The game runs.
   a) Prove that $P(E_1) = \frac{1}{6}$ and $P(N/E_1) = \frac{3}{5}$. Deduce $P(N \cap E_1)$.
   b) Prove that the probability to select two balls with different colors is $\frac{23}{50}$.
   c) Knowing that the selected balls have different colors, calculate the probability that the die shows the number 1.
2) In this part, the player plays the game for two independent times. So as the urns are back to their original states after the first round.
   In each round, the player wins \$20 if he selects two balls with different colors, otherwise he will pay \$5.
   Let X be the random variable that represents the algebraic gain of the player at the end of the two rounds.
   a) Prove that the possible values of X are 40, 15 and -10.
   b) Show that $P(X = 15) = \frac{621}{1250}$.
   c) Determine the probability distribution of X.
3) The player repeats the game for n times according to the conditions of question 2; $n \in \mathbb{N} - \{0\}$.
   Denote by P_n the probability that the player wins \$20 at least one time at the end of n rounds.
   Let (P_n) be the sequence with general term P_n.
   a) Prove that $P_n = 1 - \left(\frac{27}{50}\right)^n$.
   b) Prove that the sequence P_n is strictly increasing.
   c) Determine $\lim\limits_{n \to +\infty} (P_n)$ and interpret its value.

IV- (4,5 points)

In the next figure,
ABCD and *AHFE* are two congruent rectangles:
D, *A* and *E* are collinear
B, *A* and *H* are collinear
AB = A*E* = 1 and *AD* = *AH* = 2.
Denote by *S* the similitude that maps *A* onto *B* and *D* onto *A*.



1) Determine an angle of *S*, and calculate the ratio of *S*.
2) Consider the rotation $R\left(A;\frac{\pi}{2}\right)$ and let *I* be the intersection point of (BD) and (AF).
   a) Determine $R(D)$.
   b) Construct $L = R(B)$.
   c) Prove that $\overrightarrow{FA} = \overrightarrow{HL}$ and that the two lines (AF) and (BD) are perpendicular.
3) a) Determine the image of line (BD) and that of (AF) under *S*.
   b) Deduce that *I* is the center of *S*.
4) Let *G* be the point so that *AEGB* is a square.
   Denote by J the meeting point of (DG) and (AB).
   a) Determine the nature of $S \circ R$.
   b) Prove that *J* is the center of $S \circ R$.
5) The complex plane is referred to the system $(A;\vec{u},\vec{v})$, with $\vec{u} = \overrightarrow{AE}$ and $\vec{v} = \overrightarrow{AB}$.
   a) Write the complex form of *S*. Deduce Z_1.
   b) Determine the preimage of *G* under *S*.
   c) *N* is a variable point that moves on the circle (C') with center *G* and radius 1. Prove that the preimage of *N* moves on a circle (C) with center and radius to be determined.

V- (3 points)

In the plane referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$, consider the points $A(4;4)$ and $L(-1;4)$.

Let (*P*) the parabola with vertex *O*, passing through *A* and having $(x'x)$ as focal axis.

1) a) Show that $y^2 = 4x$ is an equation of (*P*).
   b) Calculate the coordinates of *F*, the focus of (*P*) and write an equation of (*d*), the directrix of (*P*).
   c) Draw (*P*).
2) Let *J* be the midpoint of $[FL]$.
   a) What does (*AJ*) represent for the angle $\widehat{FAL}$? Deduce that (*AJ*) is tangent to (*P*).
   b) Prove that (*AJ*) intersects (*d*) at $I\left(-1;\frac{3}{2}\right)$.
3) Denote by (*Q*) the parabola with equation $y = \sqrt{2}x^2 + a$. Calculate the exact value of *a* so that (*P*) and (*Q*) are tangent to each other at a point to be determined.

4) (E) is an ellipse with equation $\frac{x^2}{2}+\frac{y^2}{8}=1$.

   a) Calculate the coordinates of G and H, meeting points of (E) and (P).

   b) Draw (E) in the same system as that of (P).

   c) Calculate the area of the domain bounded by (P) and $[GH]$.

   d) Deduce the area of the domain located inside (E) and outside the region bounded by (P) and triangle OGH.

VI- (6 points)

Part A

Consider the differential equation $y''+3y'+2y=\left(\frac{x-1}{x^2}\right)e^{-x}$.

1) Verify that the function P defined over $]0;+\infty[$ as $P(x)=e^{-x}\ln x$ is a particular solution for (E).
2) Determine the general solution of the equation $y''+3y'+2y=0$.
3) Deduce the general solution of (E) and so the particular solution whose curve passes through the points: $A\left(1;\frac{3}{e}\right)$ and $B\left(2;\frac{3+\ln 2}{e^2}\right)$.

Part B

The plane is referred to the orthonormal system $\left(O;\vec{i};\vec{j}\right)$.

Consider the function g defined over $]0;+\infty[$ as $g(x)=-3-\ln x+\frac{1}{x}$ and denote by (C_g) its representative curve in $\left(O;\vec{i};\vec{j}\right)$.

1) Discuss the variations of g and set up its table of variations.
2) Prove that the equation $g(x)=0$ has a unique solution α. Verify that $0.45 < \alpha < 0.46$.
3) Deduce according to x the sign of $g(x)$ over $]0;+\infty[$.
4) Draw the curve (C_g).

Part C

Consider the function f defined over $]0;+\infty[$ as: $f(x)=e^{-x}(3+\ln x)$and denote by (C) its representative curve in $\left(O;\vec{i};\vec{j}\right)$.

1) Determine the limits of f at the boundaries of its domain of definition. Deduce the equations of the asymptotes to (C).
2) a) For all x in $]0;+\infty[$, prove that $f'(x)=e^{-x}.g(x)$.

   b) Discuss according to x the sign of $f'(x)$, then set up the table of variations of f.
3) Show that $f(\alpha)=\frac{e^{-\alpha}}{\alpha}$, then bound $f(\alpha)$.
4) Calculate $f(e^{-3})$ and draw (C).
5) Let A be the area of the domain bounded by (C), the x-axis and the lines with equations: $x=e^{-3}$ and $x=\alpha$. Prove that $A\le e^{-\alpha}$.

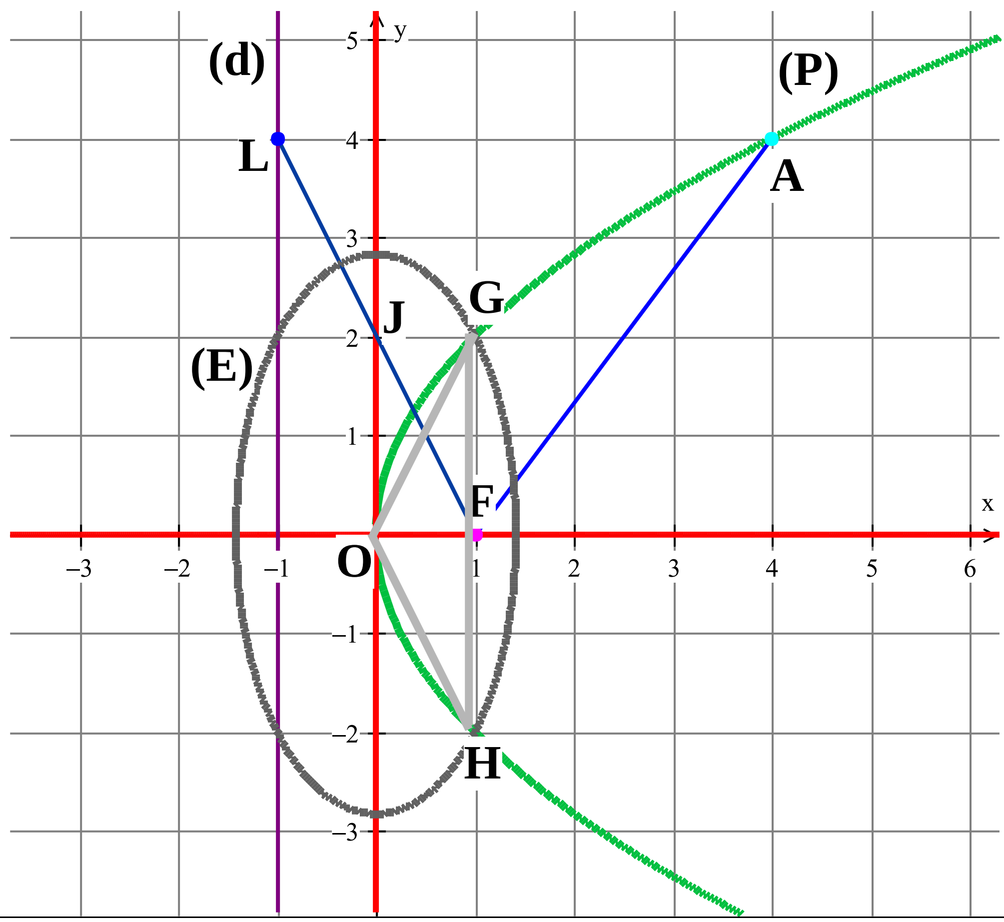
أسس التصحيح

QI	Answers	Pts
1	$\arcsin\left(\sin\left(\frac{25\pi}{4}\right)\right) = \arcsin\left(\sin\left(\frac{25\pi}{4} - 6\pi\right)\right) = \arcsin\left(\sin\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$. (b)	0.5
2	$\int \frac{1}{\sqrt{-x^2 - 4x - 3}} dx = \int \frac{1}{\sqrt{1 - (x+2)^2}} dx = \arcsin(x+2) + c$. (a)	0.5
3	a = 1 and b = 2(1+i) then $\lvert z_1 + z_2 \rvert = \left\lvert \frac{-b}{a} \right\rvert = \lvert 2(1+i) \rvert = 2\sqrt{2}$. (c)	0.5
4	$\arg\left(\frac{z + \bar{z}}{z - \bar{z}}\right) = \arg\left(\frac{2x}{2iy}\right) = \arg\left(\frac{x}{y}\right) - \arg(i) = \pi - \frac{\pi}{2} = \frac{\pi}{2}[2\pi]$; $\frac{\pi}{2}$ is an argument. (a)	0.5

QII	Answers	Pts
1a	The coordinates of A verify the two equations Or solve a system of three equations and two unknowns in *m* and *t*.	0.5
1b	The 2 lines are intersecting, they form a unique plane. The coordinates of a variable point on (d_1) and (d_2) satisfy the equation of *(P)*. **Or** $\vec{AM}.(\vec{V}_{(d_1)} \wedge \vec{V}_{(d2)}) = 0$ with M(x;y;z) is a point of (P)	1
2a	$\vec{N}_{(P)}.\vec{N}_{(Q)} = 0$	0.25
2b	(P) is perpendicular to (Q) then (P) and (Q) intersect and $(d_1)\cap(D) = (d_1)\cap(Q)$ at $H(2;-4;-6)$. **Or** write a system of parametric equations for (D) then find $(d_1)\cap(Q)$	0.75
2c	AG = AH, $m = 0$ or $m = 2$ then $G(-3;0;-7)$ or $G(5;2;3)$	0.5
2d	Since triangle AGH is isosceles with vertex A, then $\vec{AG} + \vec{AH}$ is a direction vector for a bisector of the angle $\widehat{GAH}$. $\vec{AG} + \vec{AH} \begin{vmatrix} -3 \\ -6 \\ -9 \end{vmatrix}$ or $\vec{AG} + \vec{AH} \begin{vmatrix} 5 \\ -4 \\ 1 \end{vmatrix}$	1

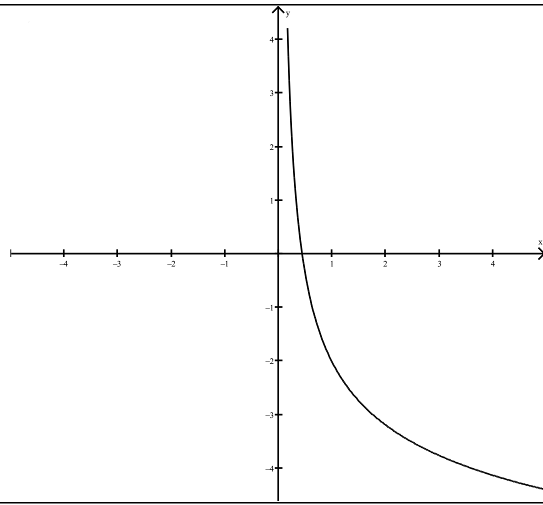
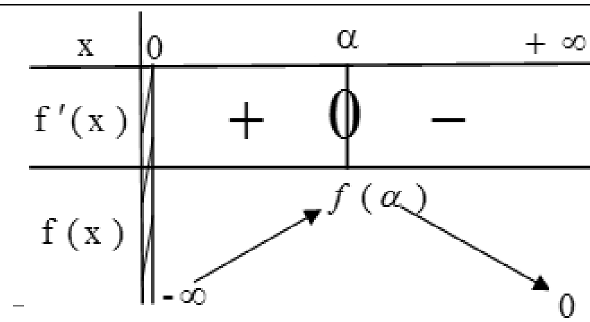
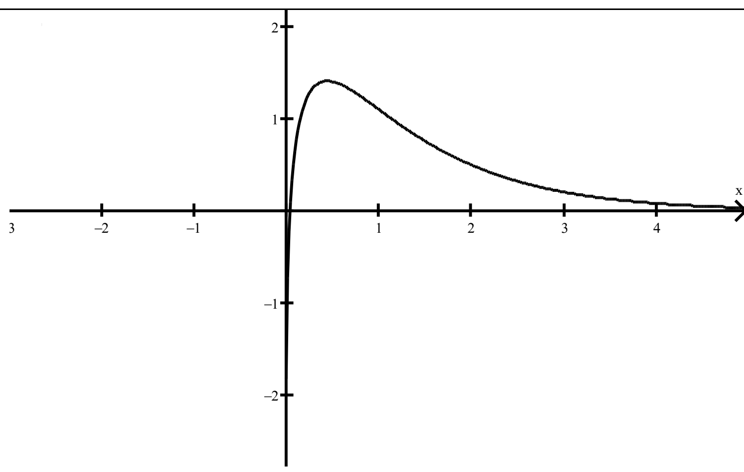
QIII	Answers	Pts
1a	$P(E_1)=P(\{1\})=\frac{1}{6}$; $P(N/E_1)=\frac{3\times 2}{C_5^2}=\frac{3}{5}$ then $P(N\cap E_1)=\frac{3}{5}\times\frac{1}{6}=\frac{1}{10}$	0.25 0.5 0.25
1b	$P(N)=\frac{1}{10}+\frac{12}{25}\times\frac{2}{6}+\frac{2}{5}\times\frac{3}{6}=\frac{23}{50}$	1
1c	$P(E_1/N)=\frac{P(E_1\cap N)}{P(N)}=\frac{1}{10}\times\frac{50}{23}=\frac{5}{23}$	0.5
2a	40 : The player obtains two balls with different color twice. 15 : The player obtains two balls with different color once. -10 : The player obtains two balls with same color twice.	0.5
2b	$P(X=15)=\frac{23}{50}\left(1-\frac{23}{50}\right)\times 2!=\frac{621}{1250}$.	0.75
2c	$P(X=15)=\frac{621}{1250}$; $P(X=40)=\frac{23}{50}\times\frac{23}{50}=\frac{529}{2500}$ and $P(X=-10)=\frac{27}{50}\times\frac{27}{50}=\frac{729}{2500}$	0.5 0.5
3a	$P_n=1-\underbrace{\left(1-\frac{23}{50}\right)\times\left(1-\frac{23}{50}\right)\times\ldots\left(1-\frac{23}{50}\right)}_{\text{n fois}}=1-\left(\frac{27}{50}\right)^n$.	0.75
3b	$P_{n+1}-P_n=1-\left(\frac{27}{50}\right)^{n+1}-\left(1-\left(\frac{27}{50}\right)^n\right)=-\left(\frac{27}{50}\right)^n\left(\frac{27}{50}\right)^1+\left(\frac{27}{50}\right)^n=$ $\left(\frac{27}{50}\right)^n\left(-\frac{27}{50}+1\right)=\left(\frac{27}{50}\right)^n\left(\frac{23}{50}\right)>0$ Then the sequence (P_n) is strictly increasing.	1
3c	$\lim\limits_{n\to+\infty}P(n)=1-0=1$ since $-1<\frac{27}{50}<1$ If n will be infinitely large, then the player will win for sure at least 20$ at the end of n rounds.	0.25 0.25

QIV	Answers	Pts
1	$(\overrightarrow{AD},\overrightarrow{BA}) = \frac{\pi}{2}[2\pi]$ then $\alpha = \frac{\pi}{2}$; $k = \frac{BA}{AD} = \frac{1}{2}$	0.5 0.5
2a	R(D) = H because AH = AD and $(\overrightarrow{AD},\overrightarrow{AH}) = \frac{\pi}{2}[2\pi]$.	0.5
2b	L is midpoint on [AD].	0.5
2c	AL = AB = HF because two triangles are congruent and (AL)↗↗ (HF) then $\overrightarrow{AL} = \overrightarrow{FH}$ so ALHF is a parallelogram hence $\overrightarrow{FA} = \overrightarrow{HL}$. R(B) = L and R(D) = H then (HL) and (DB) are perpendicular but (FA) is parallel to (HL) so (AF) and (BD) are perpendicular.	0.5 0.5
3a	Since S(D) =A then S(BD) is a line through A and ⊥ to (BD) then S(BD) = (AF) S(A) = B, then S(AF) is a line through B and ⊥to (AF) then S(AF) =(BD)	0.5 0.5
3b	S((BD)∩(AF)= (AF)∩(BD) ; S(I) = I	0.5
4a	$S \circ R = S\left(I;\frac{1}{2};\frac{\pi}{2}\right) \circ R\left(A;\frac{\pi}{2}\right) = S'\left(W;\frac{1}{2};\pi\right) = h\left(W;-\frac{1}{2}\right)$. Hence it is a dilation with center W and ratio $-\frac{1}{2}$	0.75
4b	$S \circ R(A) = S(R(A)) = S(A) = B$ then $\overrightarrow{WB} = -\frac{1}{2}\overrightarrow{WA}$ A, J and B are collinear and D, J and G are collinear so by Thales: $\frac{JB}{JA} = \frac{BG}{DA} = \frac{1}{2}$ then $\overrightarrow{JB} = -\frac{1}{2}\overrightarrow{JA}$ hence W = J = center of h.	1.25
5a	$a = \frac{1}{2}e^{i\frac{\pi}{2}} = \frac{1}{2}i$ thus $z' = \frac{1}{2}iz + b$ but S(A) = B then $z_B = \frac{1}{2}iz_A + b$; $i = \frac{1}{2}i(0) + b$; b = i then $z' = \frac{1}{2}iz + i$ $\quad z_I = \frac{b}{1-a} = \frac{i}{1-\frac{1}{2}i} = \frac{2i}{2-i} \times \frac{2+i}{2+i} = -\frac{2}{5} + \frac{4}{5}i$	1 0.5
5b	Let K be the preimage of G under S then $z_G = \frac{1}{2}iz_K + i$ with $z_G = 1+i$; $1+i = \frac{1}{2}iz_K + i$ then $\frac{2}{i} = z_K$; $z_K = -2i$ Hence K=H	0.5
5c	Let M be the preimage of N under S then M moves on a circle (C) with center H and radius r, but radius of (C′) = $\frac{1}{2}$ radius of (C), then radius of (C) = 2(1) = 2. M moves on a circle (C) with center H and radius 2.	0.5

QV	Answers	Pts
1a	(P) is a parabola with vertex O and x’x focal axis, then $y^2 = 2px$ but (P) passes through A then 16=2p(4) ; p = 2 then $y^2 = 4x$ is an equation of the parabola (P).	0.5
1b	Focus F($\frac{p}{2}$;0) then F(1 ;0) ; directrix (d) : x = $-\frac{p}{2} = -1$.	0.5
1c		0.5
2a	A is on (P) then AF = AL since F is the focus of (P) and L is the orthogonal projection of A on (d) then AFL is an isosceles triangle with vertex A. J is the midpoint [FL] then the median (AJ) ia an interior bisector of $F\hat{A}L$, thus (AJ) is tangent at A to (P).	1
2b	$\overrightarrow{AJ}\begin{vmatrix}-4\\-2\end{vmatrix}$ and $\overrightarrow{AI}\begin{vmatrix}-5\\-\frac{5}{2}\end{vmatrix}$ then the two vectors are collinear but I is a point on (d) then (AI) intersects (d) at I. **Or** write an equation of (AJ) and determine the intersection between (AJ) and (d).	0.5
3	There is a common tangent to (P) and (P′). Derive with respect to x : $2yy' = 4$ then $yy' = 2$. $y' = 2\sqrt{2}x$	1

	$y' = y'$ so $\frac{2}{y} = 2\sqrt{2}x$ then $y = \frac{1}{\sqrt{2x}}$ consequently $\left(\frac{1}{\sqrt{2x}}\right)^2 = 4x$; $x = \frac{1}{2}$ then $y = \frac{1}{\sqrt{2\left(\frac{1}{2}\right)}} = \sqrt{2}$ thus $\sqrt{2} = \sqrt{2}\left(\frac{1}{2}\right)^2 + a$ hence $a = \frac{3}{4}\sqrt{2}$	
4a	$\frac{x^2}{2} + \frac{y^2}{8} = 1$ then $y^2 = 8 - 4x^2$ then $4x = 8 - 4x^2$ consequently $x' = 1$ accepted because $1 \in \left[-\sqrt{2};\sqrt{2}\right]$ and $x'' =$ -2 rejected because $-2 \notin \left[-\sqrt{2};\sqrt{2}\right]$ Then G(1 ; 2) and H(1 ; -2).	0.5
4b	See the figure	0.5
4c	$A = 2\int_0^1 2\sqrt{x}dx = 4\left(\frac{2}{3}x\sqrt{x}\right)_0^1 = \frac{8}{3}$ u.a	0.5
4d	Area = $\pi ab - \left[A - \frac{1\times 4}{2}\right] = \pi(\sqrt{2})(2\sqrt{2}) - \frac{8}{3} + 2 = 4\pi - \frac{2}{3}$ *unit of .area*	0.5

QVI	Answers	Pts
A1	$P''(x) + 3p'(x) + 2p(x) = \left(\frac{x-1}{x^2}\right)e^{-x}$	1
A2	$y_1 = c_1e^{-x} + c_2e^{-2x}$	0.5
A3	$y = y_1 + p(x) = c_1e^{-x} + c_2e^{-2x} + e^{-x}\ln x$; $c_1 = 3$ and $c_2 = 0$; $y = e^{-x}(3 + \ln x)$	1.5
B1	$\lim_{x\to 0^+} g(x) = +\infty$; $\lim_{x\to +\infty} g(x) = -\infty$ $g'(x) < 0$ $\begin{array}{c\|c\|ccc} x & 0 & & & +\infty \\ \hline g'(x) & \| & & - & \\ \hline g(x) & \| & +\infty & \searrow & -\infty \end{array}$	1
B2	g is continuous and strictly decreasing over $]0;+\infty[$ from +ve to -ve then $g(x) = 0$ admits a unique solution α; $g(0.45) > 0$ and $g(0.46) < 0$, thus $0.45 < \alpha < 0.46$	1
B3	If $x \in]0;\alpha[$, $g(x) > 0$; if x = α, $g(x) = 0$ and if $x \in]\alpha;+\infty[$, $g(x) < 0$	0.5

B4		0.5
C1	$\lim_{x\to 0^+} f(x) = -\infty$; $\lim_{x\to +\infty} f(x) = 0^+$; $x = 0$ VA and $y = 0$ H.A	1
C2a	$f'(x) = -e^{-x}(3+\ln x) + \left(\frac{1}{x}\right)e^{-x} = e^{-x}.g(x)$	0.5
C2b	$e^{-x} > 0$ then $f'(x)$ has the same sign as $g(x)$ if $x \in]0;\alpha[$ $f'(x) > 0$; if $x = \alpha$, $f'(x) = 0$ and if $x \in]\alpha;+\infty[$ $f'(x) < 0$ 	1.5
C3	$\ln(\alpha) = -3 + \frac{1}{\alpha}$; $f(\alpha) = e^{-\alpha}\left(3 - 3 + \frac{1}{\alpha}\right) = \frac{e^{-\alpha}}{\alpha}$.; $1.37 < f(\alpha) < 1.41$	1
C4	 $f(e^{-3}) = 0$	1
C5	$A = \int_{e^{-3}}^{\alpha} f(x)dx \leq$ Area of the rectangle with dimensions α et $f(\alpha)$ then $A \leq \alpha \times f(\alpha)$; $A \leq \alpha \times \frac{e^{-\alpha}}{\alpha}$, Then $A \leq e^{-\alpha}$.	1