عدد المسائل: ست	مسابقة في مادة الرياضيات المدة: أربع ساعات	الاسم: الرقم:

ملاحظة: - يسمح باستعمال آلة حاسبة غير قابلة للبرمجة او اختزان المعلومات او رسم البيانات.
- يستطيع المرشّح الإجابة بالترتيب الذي يناسبه (دون الالتزام بترتيب المسائل الواردة في المسابقة).

I- (2.5 points)

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the two straight lines (D) and (D') defined as:

$$(D):\begin{cases} x=\lambda+1 \\ y=0 \\ z=\lambda+3 \end{cases} \quad (\lambda\in\mathbb{R}) \quad \text{and} \quad (D'):\begin{cases} x=t \\ y=3t-3 \\ z=t \end{cases} \quad (t\in\mathbb{R}).$$

1) Prove that (D) and (D') are skew (non-coplanar).
2) Denote by (P) the plane containing (D') and parallel to (D).
   Show that an equation of (P) is: $x-z=0$.
3) Write an equation of the plane (Q) containing (D) and perpendicular to (P).
4) Verify that A(1 , 0 , 1) is the point of intersection of (D') and (Q).
5) a- Determine the coordinates of point B the orthogonal projection of A on (D).
   b- Let C(1 , 0 , 3) be a point on (D).
   Verify that the triangle ABC is right isosceles.
6) Determine the coordinates of the points M on (D') so that the volume of the tetrahedron MABC is equal to 2 cubic units.

II- (2 points)

Consider the sequence (I_n) defined, for all integers $n\geq 1$, as $I_n=\int_1^e \frac{(\ln x)^n}{x^2}\,dx$.

1) Prove that $I_n\geq 0$.
2) Show that $I_{n+1}\leq I_n$ and deduce the sense of variations of (I_n).
3) Justify that the sequence (I_n) is convergent.
4) Using integration by parts, prove that: $I_{n+1}=-\frac{1}{e}+(n+1)I_n$.
5) a- Using parts 2) and 4), prove that $I_n\leq\frac{1}{ne}$.
   b- Determine $\lim\limits_{n\to+\infty} I_n$.

III- (3 points)

The plane is referred to an orthonormal system $(O ; \vec{i}, \vec{j})$.

(E) is the ellipse with equation $5x^2 + 9y^2 = 45$.

(P) is the parabola with focus $F(-2 , 0)$ and directrix (d) with equation $x = -4$.

1) Verify that an equation of (P) is $y^2 = 4x + 12$.
2) For $x \geq -3$, calculate the coordinates of the point of intersection of (E) and (P).
3) a- Determine the coordinates of the four vertices of (E).

   b- $F(-2 , 0)$ is one of the foci of (E).

   Write an equation of the directrix (Δ) of (E) associated with F.
4) Specify the common points of (P) and the line with equation x = 1, then draw (E) and (P).
5) Let $M(\alpha , \beta)$ be a point on (E).

   a- Write, in terms of α and β, an equation of the tangent (T) to (E) at M.

   b- Determine the coordinates of points M so that (T) passes through the point $K\left(\frac{9}{2}, 0\right)$.
6) Denote by (D) the line passing through F and parallel to the y-axis.

   (D) intersects (E) at A and (D) intersects (P) at B $(y_A > 0 \text{ and } y_B > 0)$.

   H is the orthogonal projection of B on (d) and F' is the second focus of (E).

   Show that $AF' - AB = 4$.

IV- (2.5 points)

Consider the three urns U, V, and W such that:

- U contains three balls numbered 1, 2 and 3.
- V contains three balls numbered 1, 2 and 3.
- W contains seven balls: three red balls and four blue balls.

Part A

One ball is randomly selected from U and one ball is randomly selected from V.

Denote by X the random variable that is equal to the absolute value of the difference of the two numbers carried by the two selected balls.

1) Verify that the possible values of X are 0, 1 and 2.
2) Prove that the probability $P(X = 2) = \frac{2}{9}$.
3) Determine the probability distribution of X.

Part B

One ball is randomly selected from U and one ball is randomly selected from V.

If the absolute value of the difference of the two numbers carried by the two selected balls is 2, then three balls are randomly and simultaneously selected from W; otherwise, three balls are randomly and successively selected with replacement from W.

Consider the events:

E: "The absolute value of the difference of the two numbers carried by the two selected balls from U and V is 2"

F: "The three balls selected from W are all red"

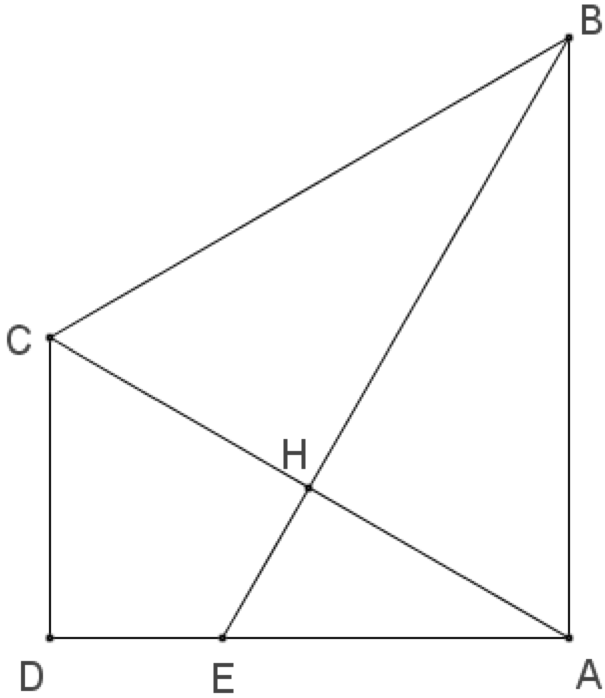
1) Show that $P(F/E) = \frac{1}{35}$, then calculate $P(F \cap E)$.
2) Prove that $P(F) = \frac{149}{2205}$.
3) Knowing that at least one of the three balls selected from W is blue, calculate the probability that the absolute value of the difference of the two numbers carried by the two selected balls from U and V is 2.

V- (3 points)

In the figure below:

- ABCD is a right trapezoid such that: $\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\left(\overrightarrow{DA};\overrightarrow{DC}\right)=\frac{\pi}{2}\quad[2\pi]$
- ABC is a direct equilateral triangle with side 2
- H is the midpoint of [AC]
- E is the point of intersection of the two lines (BH) and (AD).

Let S be the direct plane similitude that maps B onto A and A onto E.



1) a- Prove that $\frac{\sqrt{3}}{3}$ is the ratio (scale factor) of S. (You may use $\tan \widehat{EBA}$)

   b- Verify that $-\frac{\pi}{2}$ is an angle of S.

2) a- Verify that the image of (BE) under S is (AC), then determine the image of the line (AC) under S.

   b- Deduce that H is the center of S.

3) Let (Δ) be the line drawn through E and perpendicular to (AD).

   (Δ) intersects (AC) at F.

   The parallel through C to (AD) intersects (Δ) at L.

   Show that $S(E)=F$ and that $S(D)=L$.

4) Consider the direct plane similitude S' with center B, ratio $\frac{\sqrt{3}}{2}$ and an angle $\frac{\pi}{6}$.

   a- Determine the ratio and an angle of $S\circ S'$.

   b- Determine $S\circ S'(B)$.

   c- Prove that E is the center of $S\circ S'$.

VI- (7 points)

Consider the differential equation (E): $y' - 2y = 2e^{2x} - 2$.

Part A

Let $y = z + 2xe^{2x} + 1$

1) Form the differential equation (E_1) satisfied by z.
2) Solve (E_1), then deduce the general solution of (E).
3) Determine the particular solution of (E) so that $y(0) = 0$.

Part B

Consider the function g defined, on ℝ, as $g(x) = (2x-1)e^{2x} + 1$.

1) Calculate $g'(x)$ and set up the table of variations of g. (It is not required to find the limits of g at $-\infty$ and at $+\infty$).
2) Deduce the sign of g(x).

Part C

Consider the function f defined, on ℝ, as $f(x) = \begin{cases} \dfrac{e^{2x}-1}{x} & \text{for } x \neq 0 \\ 2 & \text{for } x = 0 \end{cases}$.

Denote by (C) the representative curve of f in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{x\to 0} f(x)$ and deduce that f is continuous at x = 0.
2) Determine $\lim\limits_{x\to -\infty} f(x)$ and deduce an asymptote to (C).
3) Determine $\lim\limits_{x\to +\infty} f(x)$ and $\lim\limits_{x\to +\infty} \dfrac{f(x)}{x}$.
4) a- Determine $\lim\limits_{x\to 0} \dfrac{f(x)-2}{x}$.

   b- Deduce that the line (T) with equation y = 2x + 2 is tangent to (C) at the point with abscissa 0.
5) a- Verify that $f'(x) = \dfrac{g(x)}{x^2}$ for all $x \neq 0$.

   b- Set up the table of variations of f on ℝ.
6) Draw (T) and (C).
7) a- Show that, on ℝ, f has an inverse function h whose domain of definition is to be determined.

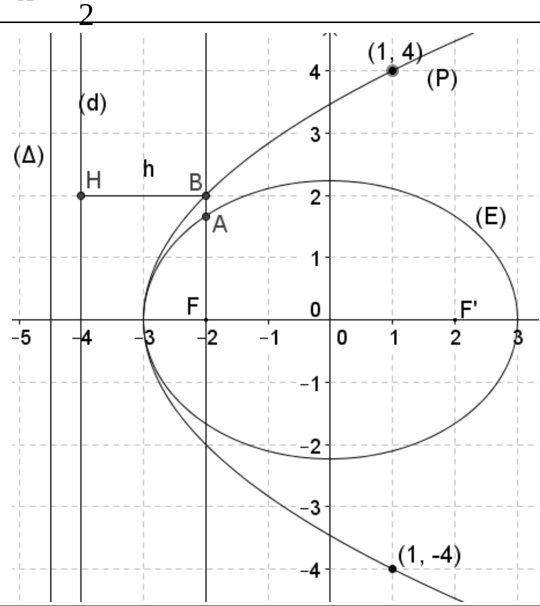
   b- Draw (C'), the representative curve of h in the same system as (C).
8) (L) is the curve with equation $y = \dfrac{1}{x}$.

   Calculate the coordinates of the point of intersection of (L) and (C').

عدد المسائل: ست — أسس تصحيح مسابقة الرياضيات

QI	Answers	M
1	$\vec{V}(1;0;1)$ is a direction vector of (D) and $\vec{V'}(1;3;1)$ is a direction vector of (D'). $I(1;0;3)\in(D)$ and $J(0;-3;0)\in(D')$. $\vec{IJ}.(\vec{V}\wedge\vec{V'})=\begin{vmatrix}-1 & -3 & -3\\ 1 & 0 & 1\\ 1 & 3 & 1\end{vmatrix}=-6$ Hence the two lines are skew.	1
2	$\vec{JM}.(\vec{V}\wedge\vec{V'})=0\Leftrightarrow\begin{vmatrix}x & y+3 & z\\ 1 & 0 & 1\\ 1 & 3 & 1\end{vmatrix}=0$ then $x-z=0$	1
3	$\vec{IM}.(\vec{V}\wedge\vec{N_P})=0\Leftrightarrow\begin{vmatrix}x-1 & y & z-3\\ 1 & 0 & 1\\ 1 & 0 & -1\end{vmatrix}=0$ then $(Q): y=0$	1
4	Substitute (D') in (Q). $3t-3=0\Leftrightarrow t=1$. Hence A(1 ; 0 ; 1).	0.5
5a	$\vec{AB}(\lambda,0,\lambda+2)$ where $\vec{AB}.\vec{V}=0$ hence $\lambda=-1$ and $B(0,0,2)$	0.5
5b	$\vec{AB}(-1,0,1);\vec{BC}(1,0,1)$ but $\vec{AB}.\vec{BC}=0$ and $AB=BC$ then ABC is a right isosceles triangle.	0.5
6	$v=\frac{1}{3}\times\text{distance(M to Q)}\times\text{Area}(\Delta ABC)$ then $2=\frac{1}{3}\lvert 3t-3\rvert\times1$ then $t=3$ or $t=-1$ $M(-1,-6,-1)$ or $M(3,6,3)$ **OR**: $\frac{1}{6}\left\lvert\vec{AM}.(\vec{AB}\wedge\vec{AC})\right\rvert=2$ …..	0.5

QII	Answers	M
1	$1\le x\le e, 0\le\ln x\le1$ and $\frac{(\ln x)^n}{x^2}\ge0$ then $I_n\ge0$.	1
2	$I_{n+1}-I_n=\int_1^e\frac{(\ln x)^n(\ln x-1)}{x^2}dx<0$	1
3	(I_n) is decreasing and bounded below ,then its convergent	0.5
4	$I_{n+1}=\int_1^e\frac{(\ln x)^{n+1}}{x^2}dx$ let $u'=\frac{1}{x^2}$ and $v=(\ln x)^{n+1}$; $I_{n+1}=\left[-\frac{1}{x}(\ln x)^{n+1}\right]_1^e+(n+1)\int_1^e\frac{(\ln x)^n}{x^2}dx$	0.5
5-a	$I_{n+1}=-\frac{1}{e}+(n+1)I_n\le I_n$ then $I_n\le\frac{1}{ne}$	0.5
5-b	$0\le I_n\le\frac{1}{ne}$ then $\lim_{n\to+\infty}I_n=0$	0.5

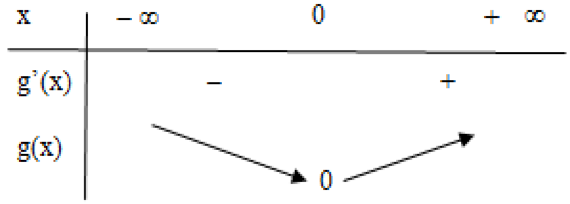
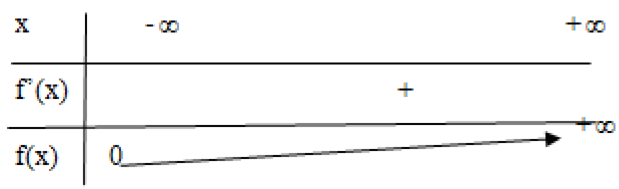
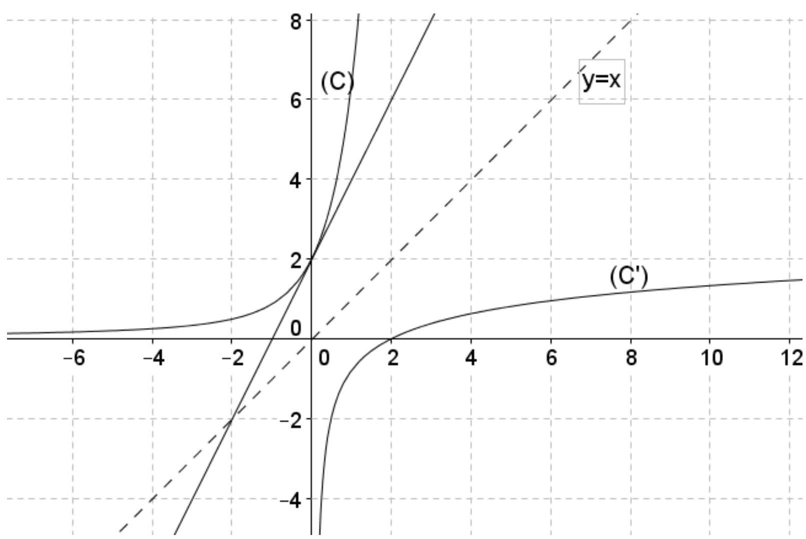
QIII	Answers	M
1	$\dfrac{MF}{d(M \to (d))} = 1$ then $(x+2)^2 + y^2 = (x+4)^2$ so $y^2 = 4x+12$	1
2	$5x^2 + 9(4x+12) = 45$ then $x' = \dfrac{-21}{5}$ rejected or $x'' = -3$ accepted. The point is: (-3,0)	1
3-a	The vertices of (E) are $(-3,0),(3,0),\left(0,\sqrt{5}\right),\left(0,-\sqrt{5}\right)$	1
3-b	$a=3, b=\sqrt{5}, c=2$ with O(0,0) being the center, so $F(-2,0)$. The associated directrix to F is (Δ) with equation $x = \dfrac{-9}{2}$	0.5
4		1
5-a	The tangent to (E) at M is (T): $\dfrac{\alpha x}{9} + \dfrac{\beta y}{5} = 1$	0.5
5-b	Substitute the coordinates $\left(\dfrac{9}{2}, 0\right)$ in the equation of (T) then $\alpha = 2$ and $\beta = \dfrac{5}{3}$, or $\beta = \dfrac{-5}{3}$ hence $M(2, \dfrac{5}{3})$ or $M(2, \dfrac{-5}{3})$	0.5
6	B is on the parabola then BF=BH, A is on the ellipse then $AF + AF' = 6$ $AF' - AB = 6 - AF - (BH - AF)$ so $AF' - AB = 4$	0.5

<table>
<tr><th>QIV</th><th>Answers</th><th>M</th></tr>
<tr><td>A1</td><td>$X(\Omega)$={ 0, 1, 2, 3}
X=0: (1,1), (2,2), (3,3)
X=1: (2,1), (1,2), (3,2), (2,3)
X=2: (3,1), (1,3)
<table>
<tr><td></td><td>1</td><td>2</td><td>3</td></tr>
<tr><td>1</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>2</td><td>1</td><td>0</td><td>1</td></tr>
<tr><td>3</td><td>2</td><td>1</td><td>0</td></tr>
</table></td><td>0.5</td></tr>
<tr><td>A2</td><td>$P(X=2) = \dfrac{2}{9}$</td><td>1</td></tr>
</table>

A3	x_i: 0, 1, 2, Total; p_i: $\frac{1}{3}$, $\frac{4}{9}$, $\frac{2}{9}$, 1	1
B1	$P\left(F/E\right)=\frac{C_3^3}{C_7^3}=\frac{1}{35}$; $P\left(F\cap E\right)=\frac{2}{9}\times\frac{1}{35}$	1
B2	$P(F)=\frac{2}{315}+\frac{7}{9}\times\left(\frac{3}{7}\right)^3=\frac{149}{2205}$	1
B3	$P\left(E/\overline{F}\right)=\frac{P(E\cap\overline{F})}{P(\overline{F})}=\frac{\frac{2}{9}\times\frac{34}{35}}{1-\frac{149}{2205}}=\frac{119}{514}$	0.5

QV	Answers	M
1-a	$k=\frac{AE}{AB}=\tan EBA=\tan 30^0=\frac{\sqrt{3}}{3}$	0.5
1-b	$\alpha=(\overrightarrow{BA},\overrightarrow{AE})=\frac{-\pi}{2}+2k\pi$	0.5
2-a	S(BE) is a straight line perpendicular to (BE) through A, so S(BE)= (AC) S(AC) = (BE)	1
2-b	$\{H\}=(AC)\cap(BE)$ so $\{S(H)\}=(BE)\cap(AC)$ so S(H)=H. (Invariant) Hence H is the center	1
3	Triangle EHF is semi equilateral with $H=90^o$ and $E=30^o$ so $HF=\frac{\sqrt{3}}{3}HE$ and $\left(\overrightarrow{HE},\overrightarrow{HF}\right)=\frac{-\pi}{2}+2k\pi$ hence S(E)=F **OR :** $E=(BH)\cap(AD), S(E)=(AH)\cap(EF)=F$ S(A)=E , and $\left(\overrightarrow{AD},\overrightarrow{EL}\right)=\frac{-\pi}{2}$ and $\frac{EL}{AD}=\frac{CD}{AD}=\frac{\sqrt{3}}{3}$ so S(D)=L	1
4a	ratio is: $\frac{\sqrt{3}}{3}\times\frac{\sqrt{3}}{2}=\frac{1}{2}$ and an angle is $-\frac{\pi}{2}+\frac{\pi}{6}=-\frac{\pi}{3}$	0.5
4b	$S\circ S'(B)=S(B)=A$	0.5
4c	$\frac{EA}{EB}=\frac{1}{2}$ and $\left(\overrightarrow{EB},\overrightarrow{EA}\right)=-\frac{\pi}{3}$ (Triangle ABE is semi-eq). So E is the center of $S\circ S'$.	1

QVI	Answers	M
A-1	$y=z+2xe^{2x}+1$ and $y'=z'+2e^{2x}+4xe^{2x}$ .substitute inD.E: $y'-2y=2e^{2x}-2$ then (E’) : z’ -2z = 0 .	1
A-2	$z=Ce^{2x}$ and $y=Ce^{2x}+2xe^{2x}+1$.	1
A-3	y(0) = 0 . then C=-1 and $y=(2x-1)e^{2x}+1$.	0.5

B-1	$g'(x) = 2e^{2x} + 2e^{2x}(2x-1)$ $= 4xe^{2x}$ 	1.5
B-2	The minimal value of g(x) is 0 then $g(x) \geq 0$.	1
C-1	$\lim\limits_{\substack{x\to 0\\ x>0}} f(x) = \lim\limits_{\substack{x\to 0\\ x<0}} f(x) = \frac{0}{0}$ IFsoL'HR ; $\lim\limits_{\substack{x\to 0\\ x>0}} f(x) = \lim\limits_{\substack{x\to 0\\ x<0}} f(x) = 2 = f(0)$ then f is continuous at x=0	1
C-2	$\lim\limits_{x\to-\infty} f(x) = 0$ then the line y=0 (x-axis) is an asymptote to (C).	1
C-3	$\lim\limits_{x\to+\infty} f(x) = \frac{+\infty}{+\infty}$ indeterminated (HR) ; $\lim\limits_{x\to+\infty} f(x) = +\infty$ and $\lim\limits_{x\to+\infty} \frac{f(x)}{x} = +\infty$	1
C-4-a	$\lim\limits_{\substack{x\to 0\\ x>0}} \frac{f(x)-2}{x} = \lim\limits_{\substack{x\to 0\\ x<0}} \frac{f(x)-2}{x} = 2$.	0.5
C-4-b	(T): $y - 2 = 2(x-0)$ since $f'(0) = \lim\limits_{x\to 0} \frac{f(x)-2}{x}$	0.5
C-5-a	$f'(x) = \frac{2e^{2x}.x - e^{2x} + 1}{x^2} = \frac{g(x)}{x^2}$	0.5
C-5-b		1
C-6		1
C-7-a	Over $]-\infty, +\infty[$ f is continuous and strictly increasing then it admits an inverse function h .	1
C-7-b	Graph of (C') is symmetric of (C) with respect to $y = x$	0.5
C-8	$(L)\cap(C): \frac{e^{2x}-1}{x} = \frac{1}{x}$ $e^{2x} = 2$, then $2x = \ln 2, x = \frac{\ln 2}{2}$, so $(\frac{\ln 2}{2}, \frac{2}{\ln 2})$ then $(C')\bigcap(L)$ at $(\frac{2}{\ln 2}, \frac{\ln 2}{2})$	1