وزارة التربية والتعليم العالي
المديرية العامة للتربية
دائرة الامتحانات

امتحانات الشهادة الثانوية العامة
الفرع : علوم عامة

دورة سنة ٢٠٠٧ الإكمالية الإستثنائية

عدد المسائل : ست	مسابقة في مادة الرياضيات المدة أربع ساعات	الاسم: الرقم:

ملاحظة : يسمح بإستعمال آلة حاسبة غير قابلة للبرمجة او اختزان المعلومات او رسم البيانات
يستطيع المرشح الإجابة بالترتيب الذي يناسبه (دون الالتزام بترتيب المسائل الوارد في المسابقة)

I – (2 points)

The complex plane is referred to a direct orthonormal system $(O\,;\,\vec{u}\,,\,\vec{v})$.

Let z be the nonzero complex number defined by its exponential form $z = r\,e^{i\alpha}$, whose conjugate is denoted by $\bar{z}$.

Consider the points A , B and C of respective affixes $z_A = z$, $z_B = \frac{1}{z}$ and $z_C = \frac{z^2}{\bar{z}}$.

1- Determine the exponential form of each of the numbers z_B and z_C in terms of r and α .

2- Determine a measure of the angle $(\overrightarrow{OB}\,;\overrightarrow{OC})$ in terms of α . Deduce the values of α such that O , B *and* C are collinear and O belongs to $[BC]$.

3- Suppose in this part that $\alpha = \frac{\pi}{4}$.

a) Verify that $z_B \times \bar{z}_C = -1$.

b) Let D be the point of affix z_D such that $z_D = -\frac{1}{\bar{z}}$.

Calculate each of the numbers $z_B - z_D$ and $z_A - z_C$ in terms of r and prove that the straight lines (BD) and (AC) are parallel.

c) Prove that $ABDC$ is an isosceles trapezoid .

II – (3 points)

The space is referred to a direct orthonormal system $(\mathrm{O}\,;\,\vec{\mathrm{i}}\,,\,\vec{\mathrm{j}}\,,\,\vec{\mathrm{k}})$.

Consider the points A(-1 ; 2 ; 0) , B(2 ; 1 ; 0) and C(0 ; 0 ; 3).

1- Calculate the area of triangle ABC.

2- Calculate the volume of the tetrahedron OABC. Deduce the distance from O to plane (ABC).

3- a) Write an equation of the plane (ABC).

b) Show that the point $\mathrm{O'}\left(\frac{18}{23};\frac{54}{23};\frac{30}{23}\right)$ is the symmetric of O with respect to plane (ABC).

c) Calculate $\cos(\widehat{\mathrm{OAO'}})$ as well as the cosine of the angle between the line (AO) and the plane (ABC) .

4- Let J be the midpoint of [AB].

a) Verify that the plane (COJ) is the mediator plane of [AB].

b) Calculate the cosine of the acute angle between the two planes (COJ) and (xOz).

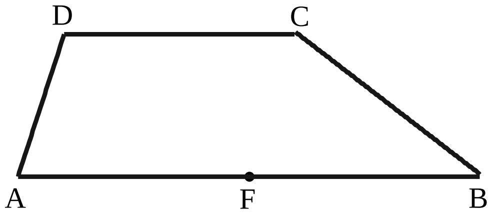
III – (2 points)

$ABCD$ is a trapezoid of bases $[AB]$ and $[CD]$ such that:

$[AB]$ is fixed and $AB = 12$;

$[CD]$ is variable and $CD = 6$.

Let F be the mid point of $[AB]$.



1- a) Prove that if the perimeter of $ABCD$ remains equal to 28, then D moves on an ellipse (E) of foci A and F.

b) Draw (E).

In all what follows, the plane is referred to the orthonormal system $(A ; \vec{i}, \vec{j})$ **such that** $B(12 ; 0)$.

2- a) Prove that $\frac{(x-3)^2}{25} + \frac{y^2}{16} = 1$ is an equation of the ellipse (E).

b) Calculate the eccentricity of (E) and determine an equation of the directrix (d) associated to A.

3- Let L be one of the points of intersection of (E) with the axis of ordinates.

a) Determine an equation of the tangent (T) to (E) at L.

b) Show that (T) cuts the focal axis of (E) at a point belonging to the directrix (d).

IV – (3 points)

In order to ensure that the cars in a given city are functioning well, a certain company is inspecting all the cars in this city.

It is known that 20 % of these cars are under guarantee.

Among the cars under guarantee, the probability that a car has a defect is $\frac{1}{100}$.

Among the cars not under guarantee, the probability that a car has a defect is $\frac{1}{10}$.

1- Calculate the probability of each of the following events:

A : « The inspected car is under guarantee and has a defect » .

D : « The inspected car has a defect » .

2- Prove that the probability that an inspected car is under guarantee knowing that it has a defect is equal to $\frac{1}{41}$.

3- The car inspection is for free if the car is under guarantee;

it costs 50 000 LL if the car is not under guarantee and does not have a defect;

it costs 150 000 LL if the car is not under guarantee and has a defect .

Denote by X the random variable that is equal to the cost of inspection of a car .

a) What are the possible values of X ?

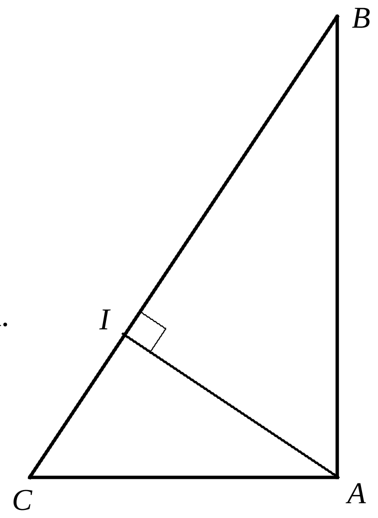
b) Determine the probability distribution of X and calculate the expected value of X .

4- The company inspects an average of 50 cars per day. Estimate the daily inspection cost collected by this company.

V – (3 points)

Given a triangle ABC such that AB = 6 , AC = 4 and $(\overrightarrow{AB}\,;\,\overrightarrow{AC}) = \frac{\pi}{2}\ (2\pi)$

Let I be the orthogonal projection of A on (BC).



1- Let h be the dilation of center I that transforms C onto B.
Construct the image (d) of the line (AC) under h.
Deduce the image D of A under h.

2- Let S be the similitude that transforms A onto B, and C onto A.
a) Determine the ratio and an angle of S.
b) Determine the image by S of each of the two straight lines (AI) and (CB). Deduce that I is the center of S.
c) Determine the image of (AB) by S.
Deduce that S(B) = D.

3- a) Determine the nature and the characteristic elements of $S \circ S$.
b) Prove that $S \circ S(A) = h(A)$.
c) Prove that $S \circ S = h$.

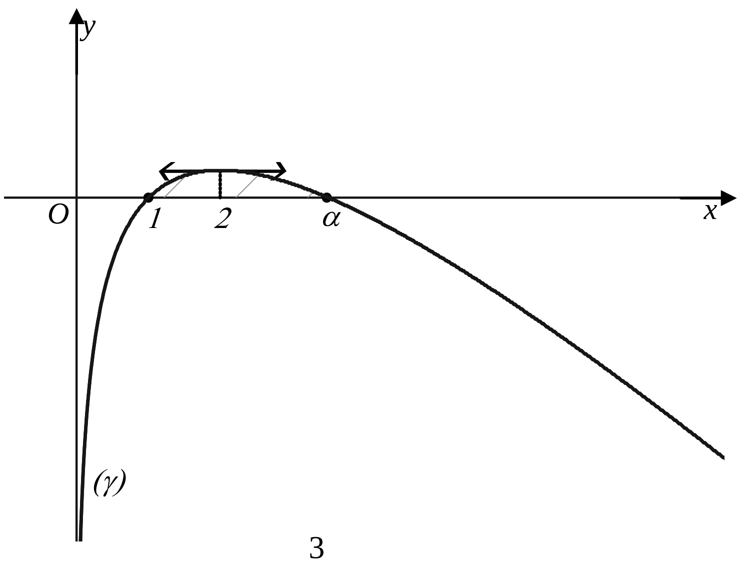
4- Let E be the mid point of [AC].
a) Determine the points F and G such that F = S(E) and G = S(F).
b) Show that the points E, I and G are collinear.

VI – (7 points)

A- Consider the differential equation (I): $xy' - y = 1 - 2\ln x$

1- Verify that $y_1 = 1 + 2\ln x$ is a particular solution of the equation (I).
2- Determine the general solution Y of the differential equation $xy' - y = 0$.
3- a) Verify that $Y + y_1$ is the general solution of the differential equation (I).
b) Determine the particular solution y of the equation (I) such that $y(1) = 0$.-

B- The figure below shows, in an orthonormal system, the representative curve (γ) of the function h defined on the interval $]0;+\infty[$ by $h(x) = 1 - x + 2\ln x$.



1- a) Prove that $3.51 < \alpha < 3.52$.

b) Determine the maximum of $h(x)$.

2- a) Using integration by parts, calculate $\int_1^{\alpha} \ln x \, dx$ in terms of α.

b) Deduce the area $S(\alpha)$ of the shaded region bounded by (γ) and the axis of abscissas.

C- Let f be the function defined on $]0\ ;+\infty[$ by $f(x) = \dfrac{1+2\ln x}{x^2}$.

Designate by (C) the representative curve of f in an orthonormal system $(O\ ;\vec{i}\ ,\vec{j})$.

1- a) Determine the point of intersection of (C) with the axis of abscissas.

b) Prove that the axes of the system are the asymptotes of (C).

2- a) Set up the table of variations of f and prove that $f(\alpha) = \dfrac{1}{\alpha}$

b) Draw (C).

3- a) Prove that the restriction of f on the interval $[1\ ;+\infty[$ has an inverse function f^{-1}.

b) Determine the domain of definition and the domain of differentiability of f^{-1}.

c) Solve the inequality $f^{-1}(x) > \alpha$.

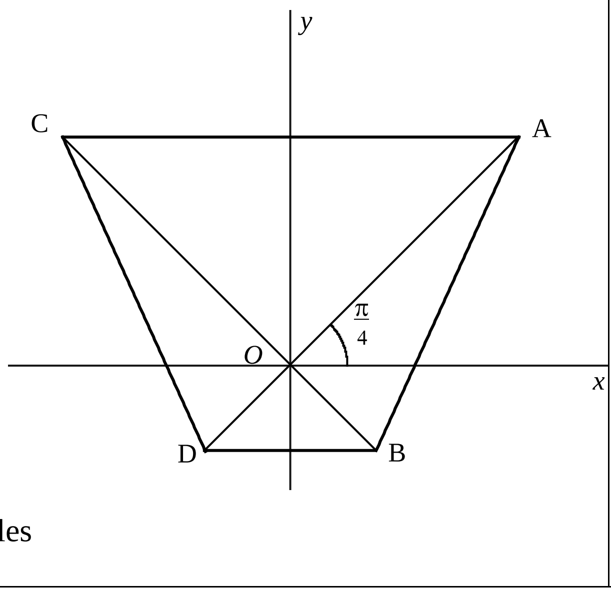
D- Let (I_n) be the sequence defined, for $n \geq 4$, by $I_n = \int_n^{n+1} f(x)\, dx$.

1-Prove that, for all x in the interval $[4\ ;+\infty[$, $0 \leq f(x) \leq \dfrac{1}{x}$.

2- Deduce that, for all natural integers $n \geq 4$, $0 \leq I_n \leq \ln\left(\dfrac{n+1}{n}\right)$.

3- Determine the limit of the sequence (I_n).

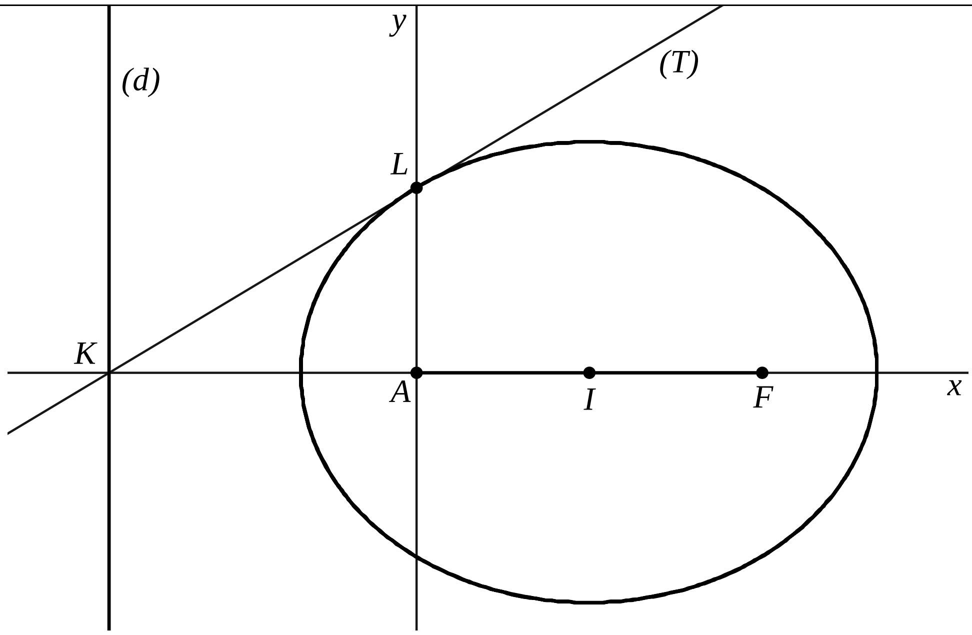
№ I- (2 points)

Part of the Q	Answer	Mark
1	$z_B = \frac{1}{z} = \frac{1}{r}e^{-i\alpha}$; $z_C = \frac{z^2}{\overline{z}} = \frac{r^2e^{i2\alpha}}{re^{-i\alpha}} = re^{i3\alpha}$.	0.5
2	$(\overrightarrow{OB}\ ;\overrightarrow{OC}) = (\vec{u}\ ;\overrightarrow{OC}) - (\vec{u}\ ;\overrightarrow{OB}) = 3\alpha - (-\alpha) = 4\alpha$. O, B and C are collinear and $O \in [BC]$ is equivalent to $(\overrightarrow{OB}\ ;\overrightarrow{OC}) = \pi + 2k\pi$ Therefore $\alpha = \frac{\pi}{4} + k\frac{\pi}{2}$ where $k \in IZ$.	0.5
3a	$z_B \times \overline{z_C} = \frac{1}{r}e^{-i\alpha} \times r\ e^{-i3\alpha} = e^{-i4\alpha} = e^{-i\pi} = -1$.	0.5
3b	$z_B - z_D = \frac{1}{z} + \frac{1}{\overline{z}} = \frac{z+\overline{z}}{z\overline{z}} = \frac{2r\cos\frac{\pi}{4}}{r^2} = \frac{\sqrt{2}}{r}$; $z_A - z_C = z - \frac{z^2}{\overline{z}} = \frac{z}{\overline{z}}(\overline{z} - z) = i(-2ir\sin\frac{\pi}{4}) = \sqrt{2}\,r$. $\frac{z_B - z_D}{z_A - z_C} = \frac{1}{r^2}$ (real number) $\frac{z_B - z_D}{z_A - z_C}$ being a real number , then (BD) and (AC) are parallel . ***or*** Each of the numbers $z_B - z_D$ and $z_A - z_C$ is a real number , then each of the straight lines (BD) and (AC) is parallel to the axis of abscissas ; consequently they are parallel	1.5
3c	$OA = OC = r$ and $OB = OD = \frac{1}{r}$. $z_D = -\frac{z}{z\overline{z}} = -\frac{z}{r^2}$. Hence, O,A and D are collinear. Therefore $ABDC$ is an isosceles trapezoid since its diagonals intersect and determine 2 isosceles triangles  	1

№ II - (3 points)

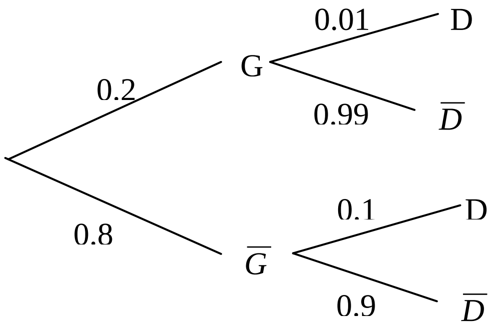
Part of the Q	Answer	Mark
1	$\overrightarrow{AB}$ (3 ; -1 ; 0) ; $\overrightarrow{AC}$ (1 ; -2 ; 3) ; $\overrightarrow{AB} \wedge \overrightarrow{AC} = -3\vec{i} - 9\vec{j} - 5\vec{k}$. The area of triangle ABC is $S = \frac{1}{2}\sqrt{9+81+25} = \frac{1}{2}\sqrt{115}$ units of area .	0.5
2	$\overrightarrow{AB} \wedge \overrightarrow{AC}(-3\,;-9\,;-5)$ and $\overrightarrow{OA}(-1\,;2\,;0)$; then $\overrightarrow{OA}.(\overrightarrow{AB} \wedge \overrightarrow{AC}) = -15$. The volume of tetrahedron $OABC$ is $V = \frac{1}{6}\left\|\overrightarrow{OA}.(\overrightarrow{AB} \wedge \overrightarrow{AC})\right\| = \frac{15}{6} = \frac{5}{2}$ units of volume. If d is the distance from O to plane (ABC) , then $V = \frac{1}{3}d \times S = \frac{d\sqrt{115}}{6}$. Therefore $d = \frac{15}{\sqrt{115}} = \frac{3\sqrt{115}}{23}$.	1
3a	(ABC) : $3x + 9y + 5z - 15 = 0$.	1
3b	$\vec{u}(3\,;9\,;5)$ is a direction vector of the straight line (OO') ; (OO') : $x = 3t$; $y = 9t$; $z = 5t$. $(OO') \cap (ABC) : t = \frac{3}{23}$; hence $(OO') \cap (ABC) = \left\{H\left(\frac{9}{23}\,;\frac{27}{23}\,;\frac{15}{23}\right)\right\}$. H being the mid point of $[OO']$; Therefore $O'\left(\frac{18}{23}\,;\frac{54}{23}\,;\frac{30}{23}\right)$.	1
3c	$\cos\ (O\hat{A}O') = \frac{\overrightarrow{AO}\ .\ \overrightarrow{AO'}}{AO \times AO'} = \frac{5}{23}$. Let α be the angle of (AO) and (ABC) . $O\hat{A}O' = 2\alpha$. $\cos^2\alpha = \frac{1+\cos 2\alpha}{2} = \frac{14}{23}$. Since α is acute , $\cos\alpha = \frac{\sqrt{322}}{23}$.	1.5
4a	$J\left(\frac{1}{2}\,;\frac{3}{2}\,;0\right)$; $\overrightarrow{OJ}.\overrightarrow{AB} = 0$ hence $(AB) \perp (OJ)$ (**or** notice that ABC is isosceles) Also $(AB) \perp (OC)$ since $\overrightarrow{OC}.\overrightarrow{AB} = 0$ (**or** notice that $(AB) \subset (xOy)$ and $C \in z'z$). Therefore the plane (COJ) is the mediator plane of $[AB]$.	0.5
4b	$\vec{j} \perp (xOz)$ and $\overrightarrow{AB} \perp (COJ)$; therefore $\cos\beta = \frac{\left\|\vec{j}.\overrightarrow{AB}\right\|}{AB} = \frac{1}{\sqrt{10}}$.	0.5

№ III- (2 points)

Part of the Q	Answer	Mark
1a	If $AB+CD+BC+DA=28$ then $12+6+DF+DA=28$. Hence $DF+DA=10>AF$. The point D varies on the ellipse (E) of foci A and F and of length of focal axis $2a=10$.	1
1b		0.5
2a	$I(3\,;\,0)$ is the center of (E) ; $a=5$ and $c=\frac{1}{2}AF=3$. Then $b=4$. The focal axis of (E) being $x'x$, then (E) : $\frac{(x-3)^2}{25}+\frac{y^2}{16}=1$.	1
2b	$e=\frac{c}{a}=\frac{3}{5}$ and (d) : $x=x_I-\frac{a^2}{c}$ = $3-\frac{25}{3}$; (d) : $x=-\frac{16}{3}$.	0.5
3a	$L(0\,;\,\frac{16}{5})$. (T) : $16(x_L-3)(x-3)+25(y_L)y=400$. (T) : $-3(x-3)+5y=25$; $-3x+5y=16$.	0.5
3b	(T) cuts $x'x$ at $K(-\frac{16}{3}\,;\,0)$ that belongs to the directrix (d) of (E) .	0.5

№ IV- (3 points)

Part of the Q	Answer	Mark
1	Let G be the event : « the car is guaranteed » We can construct the following tree :	1.5

<table>
<tr><td></td><td>


$A = G \cap D$, then p(A) = p (G) × p(D / G) = $0.2 \times 0.01 = 0.002$.

$P(D) = P(G \cap D) + P(\overline{G} \cap D) = 0.002 + 0.8 \times 0.1 = 0.002 + 0.08 = 0.082$.
</td><td></td></tr>
<tr><td>2</td><td>$p\left(G/D\right) = \frac{p(G \cap D)}{p(D)} = \frac{0.002}{0.082} = \frac{2}{82} = \frac{1}{41}$.</td><td>0.5</td></tr>
<tr><td>3a</td><td>The possible values of X are : 0 ; 50 000 and 150 000.</td><td>0.5</td></tr>
<tr><td>3b</td><td>
$p(X = 0) = p(G) = 0.2$.

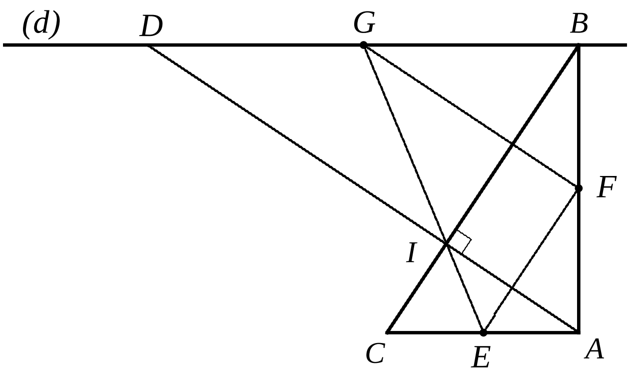
$p(X = 50\,000) = p(\overline{G} \cap \overline{D}) = P(\overline{G}) \times P(\overline{D}/\overline{G}) = 0.8 \times 0.9 = 0.72.$

$p(X = 150\,000) = p(\overline{G} \cap D) = 0.1 \times 0.8 = 0.08.$

<table>
<tr><td>x_i</td><td>0</td><td>50 000</td><td>150 000</td></tr>
<tr><td>P(X= x_i)</td><td>0.2</td><td>0.72</td><td>0.08</td></tr>
</table>

$E(X) = \sum p_i x_i = 0 + 0.72 \times 50000 + 0.08 \times 150000 = 36000 + 12000 = 48000$LL.
</td><td>2.5</td></tr>
<tr><td>4</td><td>The average control cost of a car is equal to 48 000 LL.
Therefore , if the company has controlled 50 cars , we can estimate a cost of $48000 \times 50 = 2\,400\,000$ LL.</td><td>1</td></tr>
</table>

№ V- (3 points)

<table>
<tr><th>Part of the Q</th><th>Answer</th><th>Mark</th></tr>
<tr><td>1</td><td>


$h(C) = B$.

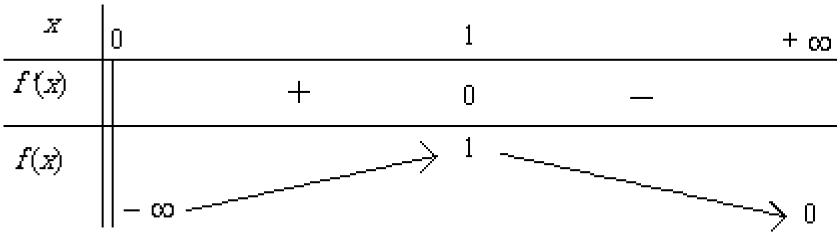
The image (d) of (AC) by h is the parallel to (AC) that passes through B.

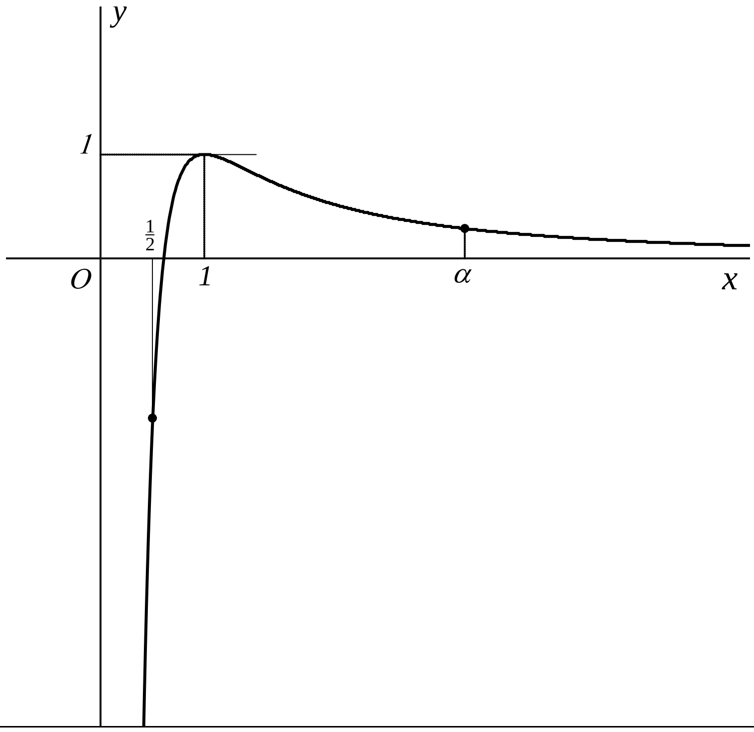
The image D of A by h is the point of intersection of (AI) and (d) .
</td><td>1</td></tr>
</table>

2a	The angle of S is $\alpha = (\overrightarrow{AC};\overrightarrow{BA}) = \frac{\pi}{2}\ [2\pi]$; its ratio is $k = \frac{BA}{AC} = \frac{3}{2}$	0.5
2b	$S(A) = B$ and $\alpha = \frac{\pi}{2}$, hence $S(AI)$ is the perpendicular to (AI) passing through B; $S(AI) = (BC)$. Similarly , $S(CB) = (AI)$. $S(AI) = (BC)$ and $I \in (AI)$ hence $S(I) \in (BC)$. $S(CB) = (AI)$ and $I \in (BC)$ hence $S(I) \in (AI)$. Therefore $S(I) = I$ and I is the center of S.	1.5
2c	$S(A) = B$ and $\alpha = \frac{\pi}{2}$, hence $S(AB)$ is the perpendicular to (AB) passing through B; $S(AB) = (d)$. $S(AB) = (d)$ and $B \in (AB)$ hence $S(B) \in (d)$; $B \in (BC)$, then $S(B) \in (AI)$,therefore $S(B) = (AI) \cap (d)$ and $S(B) = D$.	0.5
3a	$S \circ S = S(I, \frac{3}{2}, \frac{\pi}{2}) \circ S(I, \frac{3}{2}, \frac{\pi}{2}) = S(I, \frac{9}{4}, \pi)$. Hence $S \circ S$ is the dilation $h(I, -\frac{9}{4})$	0.5
3b	$S \circ S(A) = S(S(A)) = S(B) = D$.	0.5
3c	$S \circ S$ and h are two dilations of same center I and $S \circ S(A) = h(A) = D$ hence $S \circ S = h$.	0.5
4a	$S([AC]) = [BA]$ and E is the mid point of $[AC]$; hence $S(E)$ is the mid point F of $[AB]$. $S([AB]) = [BD]$ and F is the mid point of $[AB]$; hence $S(F)$ is the mid point G of $[BD]$.	0.5
4b	$G = S(F) = S \circ S(E) = h(E)$. Then E, I and G are collinear .	0.5

№ VI- (7 points)

Part of the Q	Answer	Mark
A1	$x\,y'_1 - y_1 = 2 - 1 - 2\ell n\,x = 1 - 2\ell n\,x$.	0.5
A2	$y = 0$ is a particular solution of the equation (2) : $xy' - y = 0$; If $y \neq 0$, $\frac{y'}{y} = \frac{1}{x}; \ln\lvert y \rvert = \ln x + k; y = ax$ The general solution of (2) is $Y = a\,x$ where $a \in IR$.	1
A3a	$Y + y_1 = 1 + ax + 2\ell n\,x$ depends on an arbitrary constant and satisfies the equation (1) . Therefore $y = 1 + ax + 2\ell n\,x$ is the general solution of (1) .	0.5
A3b	$y(1) = 0$ iff $a = -1$; $y = 1 - x + 2\ell n\,x$.	0.5
B1a	$h(3.51) \times h(3.52) \approx (0.001)(-0.003) < 0$. Hence $3.51 < \alpha < 3.52$.	1
B1b	The maximum of $h(x)$ is $h(2) = -1 + \ell n 4$.	0.5
B2a	$\int_1^\alpha \ell n\,x\,dx = [x\,\ell n\,x]_1^\alpha - \int_1^\alpha dx = [x\,\ell n\,x - x]_1^\alpha = \alpha\,\ell n\,\alpha - \alpha + 1$.	1

<table>
<tr><td>B2b</td><td>$S=\int_1^\alpha h(x)dx=\left[x-\frac{1}{2}x^2\right]_1^\alpha+2(\alpha\,\ell n\alpha-\alpha+1)=\frac{3}{2}+2\alpha\,\ell n\alpha-\alpha-\frac{1}{2}\alpha^2$ units of area .</td><td>0.5</td></tr>
<tr><td>C1a</td><td>$(\frac{1}{\sqrt{e}}\ ;0)$</td><td>0.5</td></tr>
<tr><td>C1b</td><td>$\lim\limits_{x\to 0^+} f(x)=-\infty$ and $\lim\limits_{x\to+\infty} f(x)=0^+$; $x'x$ and $y'y$ are asymptotes to (C) .</td><td>0.5</td></tr>
<tr><td>C2a</td><td>

$f'(x)=-\frac{4\ell n\,x}{x^3}$. $f(\alpha)=\frac{1+2\ell n\alpha}{\alpha^2}=\frac{h(\alpha)+\alpha}{\alpha^2}=\frac{1}{\alpha}$.</td><td>1,5</td></tr>
<tr><td>C2b</td><td>
</td><td>1,5</td></tr>
<tr><td>C3a</td><td>f is continuous and strictly decreasing on $[1\,;+\infty[$; f admits an inverse function f^-</td><td>0.5</td></tr>
<tr><td>C3b</td><td>f^{-1} is defined on $f([1\,;+\infty[)=]0\,;1]$.
f is differentiable on $[1\,;+\infty[$ and the equation $f'(x)=0$ admits a unique solution x
Therefore f^{-1} is differentiable on $f(]1\,;+\infty[)=]0\,;1[$</td><td>1</td></tr>
<tr><td>C3c</td><td>$f^{-1}(x)>\alpha$ is equivalent to $f(f^{-1}(x))<f(\alpha)$. Hence $x<\frac{1}{\alpha}$; i.e. $x\in]0\,;\frac{1}{\alpha}[$.</td><td>0.5</td></tr>
<tr><td>D1</td><td>The figure drawn in C2b shows that , for all $x\in[4;+\infty[$, $f(x)>0$.
In addition , $f(x)-\frac{1}{x}=\frac{1+2\ell n\,x-x}{x^2}=\frac{h(x)}{x^2}$; for $x\in[4;+\infty[$, $x>\alpha$ and $h(x)\prec 0$.
Then , for all $x\in[4;+\infty[$, $0\le f(x)\le\frac{1}{x}$.</td><td>1</td></tr>
</table>

D2	For all $n \geq 4$, $0 \leq \int_{n}^{n+1} f(x)dx \leq \int_{n}^{n+1} \frac{dx}{x}$; then $0 \leq I_n \leq \ell n\left(\frac{n+1}{n}\right)$.	1
D3	$\lim_{n\to+\infty} \ell n\left(\frac{n+1}{n}\right) = \ell n(1) = 0$; then $\lim_{n\to+\infty} I_n = 0$.	0.5