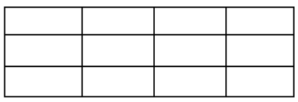
وزارة التربية والتعليم العالي المديرية العامة للتربية دائرة الامتحانات	امتحانات الشهادة الثانوية العامة الفرع: علوم عامة	دورة سنة ٢٠٠٨ الاكمالية الاستثنائية
عدد المسائل: ست	مسابقة في مادة الرياضيات المدة أربع ساعات	الاسم: الرقم:

إرشادات عامة :- يسمح باستعمال آلة حاسبة غير قابلة للبرمجة أو اختزان المعلومات او رسم البيانات
- يستطيع المرشح الإجابة بالترتيب الذي يناسبه (دون الالتزام بترتيب المسائل الوارد في المسابقة)

I- (2 points)

In the following table, only one among the proposed answers to each question is correct.
Write down the number of each question and give, **with justification,** the answer which corresponds to it.

N°	Questions	Answers a	Answers b	Answers c
1	Let $f(x)=\arctan\left(\frac{2x}{1-x^2}\right)$ for $x\in]-\infty;-1[$; then we get:	$f(x)=\pi+2\arctan(x)$	$f(x)=-2\arctan(x)$	$f(x)=\pi-2\arctan(x)$
2	$f(x)=\ln(x)$, defined on $]0;+\infty[$, the n^{th} derivative of f is given by:	$f^{(n)}(x)=\frac{(-1)^n n!}{x^n}$	$f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{x^n}$	$f^{(n)}(x)=\frac{1}{x^n}$
3	The number of rectangles in the following figure is: 	60	12	20
4	The equation $e^{2x}+2x-1=0$, has in the set IR :	2 distinct roots	No roots	One root only.
5	If $z=e^{i\frac{\pi}{2}}+e^{-i\frac{\pi}{6}}$, then:	$\arg(z)=\frac{\pi}{2}-\frac{\pi}{6}$	$\arg(z)=\frac{\pi}{2}+\frac{\pi}{6}$	$\arg(z)=\frac{\pi}{6}$.

II- (2 points)

The space is referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$.

Consider the point A(2 ; –3 ; 5) and the planes (P) and (Q) of equations:

$$(P): 2x - 2y - z + 4 = 0$$

$$(Q): 2x + y + 2z + 1 = 0$$

A-1) Show that the two planes (P) and (Q) are perpendicular.

2) Show that the straight line (D) defined by: $\begin{cases} x = t \\ y = 2t + 3 \\ z = -2t - 2 \end{cases}$ (t is a real parameter),

is the intersection of (P) and (Q).

3) Calculate the coordinates of the point H, the orthogonal projection of A on the straight line (D).

B- Designate by (R) the plane passing through the point W (1 ; 4 ; 1) and parallel to (Q).

Consider in (R) the circle (C) of center W and radius 3.

1) Find an equation of (R).

2) Show that B (3; 2; 0) is a point on (C).

3) Write a system of parametric equations of the tangent (T) at B to (C).

III-(3 points)

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Let f be the transformation that associates to each point M of affix z , the point M' of affix z' such that $z' = (\bar{z} - 2)(\bar{z} + 1)$ where $\bar{z}$ is the conjugate of z .

Designate by (x ; y) the coordinates of M and by (x' ; y') those of M' .

1) Calculate x' and y' in terms of x and y and prove that , when M' varies on the axis of ordinates, M varies on the curve (C) of equation: $x^2 - y^2 - x - 2 = 0$.

2) a- Prove that (C) is a hyperbola whose center , vertices and foci are to be determined .

b- Draw (C).

3) Let E be the point of (C) of abscissa 3 and of positive ordinate .

a- Write an equation of the tangent (t) at E to (C) .

b- The line (t) cuts the asymptotes of (C) at P and Q. Prove that E is the mid point of [PQ].

4) Designate by (D) the region limited by (C) and the line of equation x = 3.

Calculate the volume generated by the rotation of (D) about the axis of abscissas.

IV- (3 points)

An urn contains $n+10$ balls ($n \geq 2$): n white balls, 6 red balls and 4 black balls .

A- We draw simultaneously and randomly two balls from the urn.

1) Calculate the probability $q(n)$ of drawing two white balls.

2) Denote by $p(n)$ the probability of drawing two balls of the same color .

a- Prove that $p(n) = \dfrac{n^2 - n + 42}{(n+10)(n+9)}$.

b- Verify that $\lim\limits_{n\to+\infty} p(n) = \lim\limits_{n\to+\infty} q(n)$. Interpret this result.

c- Is there a case where $p(n) = \dfrac{31}{105}$?

B- Suppose in this part that $n = 3$.
A game consists of drawing simultaneously and randomly two balls from the urn.
If the two drawn balls are of the same color the player marks $+4$ points ; if not, he marks -1 point .
The player repeats the game twice by replacing , after the first draw, the drawn balls in the urn.
Let X be the random variable that is equal to the sum of points marked by the player.

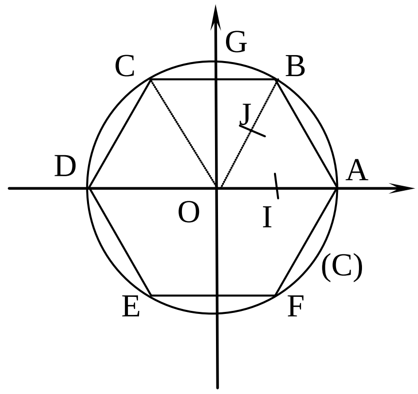
1) Justify that the values of X are : -2, 3 and 8.

2) Determine the probability distribution of X.

3) Calculate the mean (expected value) E(X).

V- (3 points)

In an oriented plane, given a direct regular hexagon ABCDEF of center O, such that:$(\overrightarrow{OA}\,;\,\overrightarrow{OB}) = \dfrac{\pi}{3}\ (2\pi)$.
(C) is the circle circumscribed about this hexagon.
I and J are the midpoints of [OA] and [OB] respectively.



Let S be the similitude that transforms A onto B and B onto J.

1)a- Determine the ratio and an angle of S.
b-Prove that S(D) = A. Find S(O) and verify that S(C) = I.
c- Determine the image of the hexagon ABCDEF by S.

2) The circle (C') is the image of (C) by S. Determine the center and the ratio of each of the two dilations (homothecies) that transforms (C) onto (C').

3) G is the midpoint of the arc BC on the circle (C).
The plane is referred to the orthonormal system $(O\,;\,\overrightarrow{OA}\,,\overrightarrow{OG})$.
a- Find the affix of each of the points B, C, E and F.
b- Write the complex form of S and deduce the affix of its center W.
c- H is the point of intersection of [AJ] and [BI]. Determine the point H' the image of H by S.

VI- (7 points)

Let f be the function defined over $I =]0;+\infty[$ by $f(x) = x^2 + \ln x$ and (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

A-1) Calculate f '(x) and determine the sense of variations of f over $]0;+\infty[$.

2) a- Calculate $\lim\limits_{x\to 0} f(x)$ and deduce an asymptote to (C).

b- Calculate $\lim\limits_{x\to +\infty} f(x)$ and $\lim\limits_{x\to +\infty} \frac{f(x)}{x}$.

c- Set up the table of variations of f.

d- Deduce that the equation $x^2 + \ln x = 0$, has a unique solution α and such that $0.6 < \alpha < 0.7$. Study the sign of f(x) according to the values of x.

3) a- Prove that (C) has a point of inflection whose abscissa is to be determined.

b- Draw (C).

4) a- Prove that f has, over I, an inverse function f^{-1} whose domain of definition is to be determined.

b- Let (C') be the representative curve of f^{-1}. Show that the point A(1;1) is common to (C) and (C') and draw (C') in the system $(O;\vec{i},\vec{j})$.

c- Write an equation of the tangent at A to (C').

d- Designate by S(α) the area of the region limited by (C) , (C') , the axis of abscissas and the axis of ordinates. Calculate S(α).

B- Let (T) be the representative curve of the function h defined over $I =]0;+\infty[$ by h(x) = lnx.

1) Study the relative positions of (C) and (T) and draw (T) in the same system as that of (C).

2) Let g be the function defined over I by $g(x) = x^2 + (\ln x)^2$.

a- Calculate g'(x) and verify that $g'(x) = \frac{2}{x} f(x)$.

b- Deduce the sense of variations of g over I.

3) Let M_0 be the point of (T) of abscissa α and M any point of (T) of abscissa x.

a- Calculate OM_0^2 in terms of α and OM^2 in terms of x.

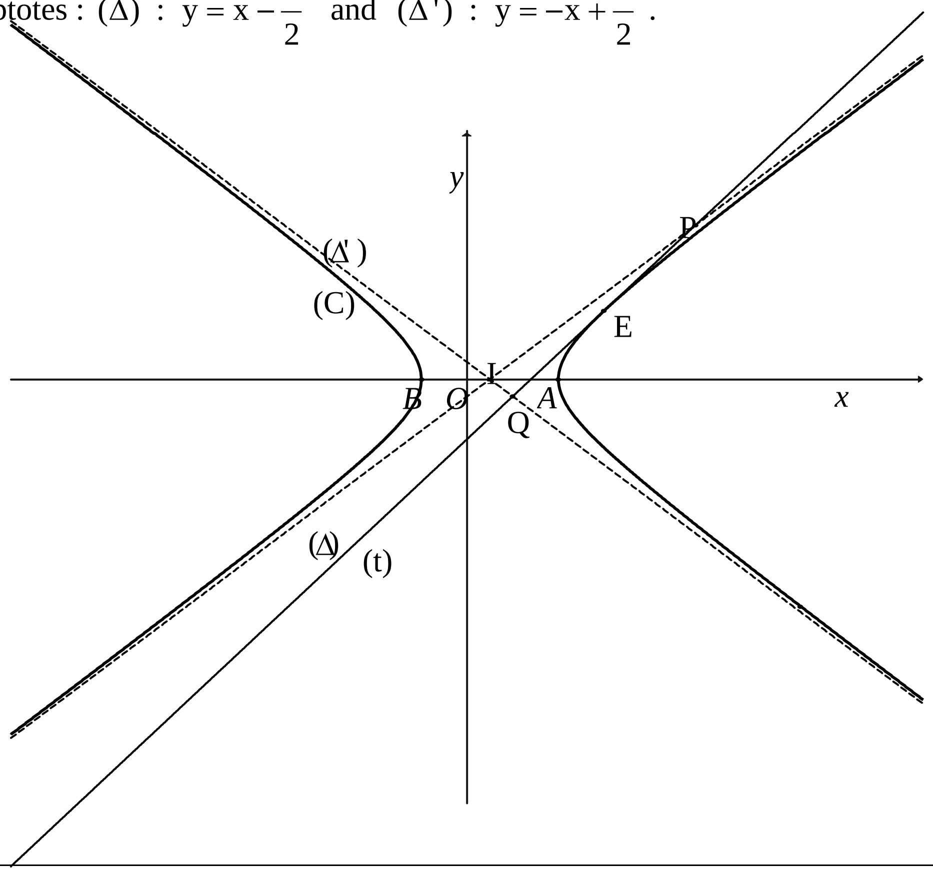
b- Prove that $OM_0 \le OM$ for all x in I.

c- Prove that the tangent at M_0 to (T) is perpendicular to (OM_0).

QI	Answer	Mark
1	$f(x)=\arctan\left(\dfrac{2x}{1-x^2}\right)$ For $x=-\sqrt{3}$, $f(x)=\pi+2\arctan(x)$ (a)	1
2	The *n*th derivative of f is: $f^{(n)}(x)=\dfrac{(-1)^{n-1}(n-1)!}{x^n}$ .Since the 2^{nd}derivative is $-1/x^2$ (b)	1
3	The number of rectangles in this figure is $C_5^2\times C_4^2=60$ (a)	0.5
4	The function $f(x)=e^{2x}+2x-1$ is continuous and strictly increasing on IR from $-\infty$ to $+\infty$, So the equation $e^{2x}+2x-1=0$ has in R a unique root. (c)	1
5	$z=e^{i\frac{\pi}{2}}+e^{-i\frac{\pi}{6}}=\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i=e^{i\frac{\pi}{6}}$. (c)	0.5

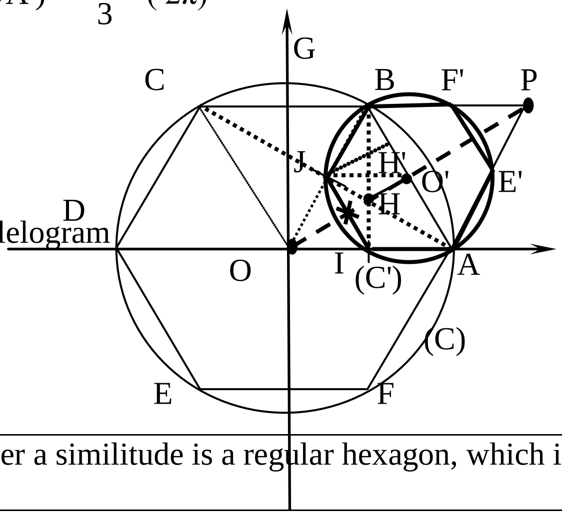
QII	Answer	Mark
A1	$\vec{N}_P(2\,;-2\,;-1)$ and $\vec{N}_Q(2\,;1\,;2)$ then $\vec{N}_P\cdot\vec{N}_Q=0$, hence (P) ⊥ (Q).	0.5
A2	For all $t\in\mathbb{R}$: $\begin{cases}2t-2(2t+3)-(-2t-2)+2=0\\2t+(2t+3)+2(-2t-2)+1=0\end{cases}$ then (D) is the line of intersection of (P) and (Q).	0.5
A3	H = proj(A/(D)) ; f(t)= $AH^2=(t-2)^2+(2t+6)^2+(2t+7)^2$ with f '(t) = 0 ; Then t = – 8/3 So H(– 8/3 ; – 7/3 ; 10/3)	1
B1	(R) // (Q) ; (R): 2x + y + 2z + r = 0 but (R) passes through W(1 ; 4 ; 1). So 2(1) + 4 + 2(1) + r = 0 ; r = – 8 ; (R) : 2x + y + 2z – 8 = 0	0.5
B2	B is a point of (R) since 2(3) + 2 + 2(0) – 8 = 0 and WB = 3 , therefore B ∈ (C).	1
B3	For all points M(x ; y ; z) of (T): $\vec{BM}$ and $(\vec{BW}\wedge\vec{N}_R)$ are collinear; $\vec{BW}\wedge\vec{N}_R(3\,;6\,;-6)$; (T) : x = k + 3 ; y = 2k + 2 , z = – 2k (k real parameter)	0.5

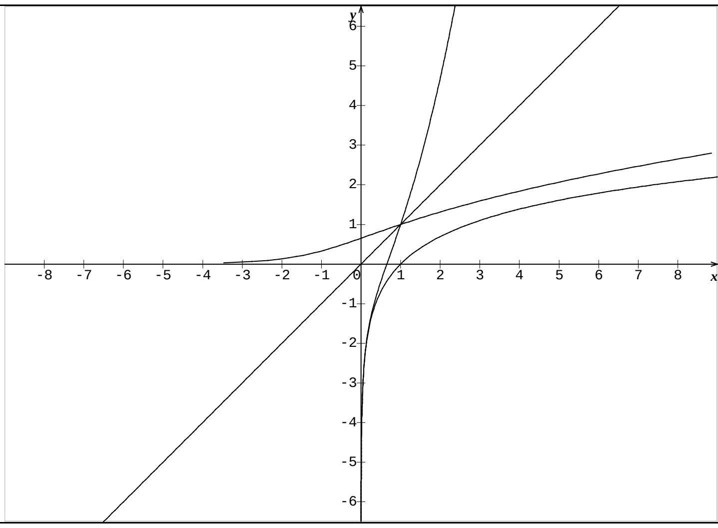
QIII	Answer	Mark
1	$z'=(\bar{z}-2)(\bar{z}+1)=\bar{z}^2-\bar{z}-2=x^2-y^2-x-2+(y-2xy)i$. Therefore $x'=x^2-y^2-x-2$ and $y'=y-2xy$. $M'\in y'y \Leftrightarrow x'=0\Leftrightarrow x^2-y^2-x-2=0$.	1
2a	$x^2-y^2-x-2=0\Leftrightarrow\left(x^2-x+\frac{1}{4}\right)-y^2-2-\frac{1}{4}=0\Leftrightarrow\left(x-\frac{1}{2}\right)^2-y^2=\frac{9}{4}$. Then (C) is a rectangular hyperbola of center $I\left(\frac{1}{2};0\right)$ and focal axis $x'x$ with $a^2=b^2=\frac{9}{4}$ and $c=a\sqrt{2}=\frac{3\sqrt{2}}{2}$.	1.5

	Vertices : $A\left(\frac{1}{2}+\frac{3}{2}\ ;\ 0\right)$ and $B\left(\frac{1}{2}-\frac{3}{2}\ ;\ 0\right)$; that is $A(2\ ;\ 0)$ and $B(-1\ ;\ 0)$. Foci : $F\left(\frac{1}{2}+\frac{3\sqrt{2}}{2}\ ;\ 0\right)$ and $F'\left(\frac{1}{2}-\frac{3\sqrt{2}}{2}\ ;\ 0\right)$.	
2b	Asymptotes : $(\Delta)\ :\ y = x-\frac{1}{2}$ and $(\Delta')\ :\ y=-x+\frac{1}{2}$. 	1
3a	E(3 ; 2) ; (t) : $(x_E-\frac{1}{2})(x-\frac{1}{2})-yy_E=\frac{9}{4}$; $y=\frac{5}{4}x-\frac{7}{4}$	0.5
3b	$(t)\cap(\Delta)$: $\frac{5x-7}{4}=x-\frac{1}{2}$;then $x_P=5$. $(t)\cap(\Delta')$: $\frac{5x-7}{4}=-x+\frac{1}{2}$;then $x_Q=1$. $\frac{x_P+x_Q}{2}=3=x_E$ and P , Q , E are collinear , then E is the mid point of [PQ].	1
4	$V=\pi\int_2^3 y^2dx=\pi\int_2^3(x^2-x-2)dx=\pi[\frac{x^3}{3}-\frac{x^2}{2}-2x]_2^3=\frac{11}{6}\pi u^3$	1

QIV	Answer	Mark
A1	$q(n)=\frac{C_n^2}{C_{n+10}^2}=\frac{n(n-1)}{(n+10)(n+9)}$.	0.5
A2a	$p(n)=\frac{C_n^2}{C_{n+10}^2}+\frac{C_6^2}{C_{n+10}^2}+\frac{C_4^2}{C_{n+10}^2}=\frac{n(n-1)+30+12}{(n+10)(n+9)}=\frac{n^2-n+42}{(n+10)(n+9)}$.	1
A2b	$\lim_{n\to+\infty} p(n)=1=\lim_{n\to+\infty} q(n)$. the number of white balls increase indefinitely , the probability of drawing two white balls is equal to the probability of drawing 2 balls of the same color and this event is almost the certain event .	1

A2c	$\frac{n^2 - n + 42}{(n+10)(n+9)} = \frac{31}{105}$; $74n^2 - 694n + 1620 = 0$; $n = 5$ or $n = 4.378$. Therefore $n = 5$ (n is a natural number greater than 2)	1
B1	The values of X are : $-1 - 1 = -2$, $4 - 1 = 3$ and $4 + 4 = 8$.	0.5
B2	p (the two drawn balls have the same color) = $p(3) = \frac{48}{13\times12} = \frac{4}{13}$. $p(X = -2) = (9/13)^2$; $p(X = 3) = 2\times 4/13 \times 9/13$ and $p(X = 8) = (4/13)^2$.	1.5
B3	$E(X) = (-2)(9/13)^2 + 2\times3\times 4/13 \times 9/13 + 8\times(4/13)^2 = 1.0769$.	0.5

QV	Answer	Mark
1a	S(A) = B and S(B) = J ; So the ratio is: $\frac{BJ}{AB} = \frac{1}{2}$ (regular hexagon). An angle is: $(\overrightarrow{AB}\ ;\ \overrightarrow{BJ}) = \frac{2\pi}{3}\ (2\pi)$	0.5
1b	We have : $\frac{BA}{AD} = \frac{1}{2}$ and $(\overrightarrow{AD}\ ;\ \overrightarrow{BA}) = \frac{2\pi}{3}\ (2\pi)$ with S(A) = B then S(D) = A. O is the midpoint of [AD] and S([AD]) = [BA] , then S(O) = O' the midpoint of [BA]. ABCO is a direct parallelogram, Then its image by S is a direct parallelogram not other than BIJO' then S(C) = I .  	1
1c	The image of a regular hexagon under a similitude is a regular hexagon, which is the hexagon BJIAE'F' * see fig.*	0.5
2	The homothecy that transforms (C) onto (C ') are: A positive dilation (homothecy) of ratio $\frac{1}{2}$ and center P such that $\overrightarrow{PO'} = \frac{1}{2}\overrightarrow{PO}$ And a negative homothecy of ratio $-\frac{1}{2}$ and center K such that $\overrightarrow{KO'} = -\frac{1}{2}\overrightarrow{KO}$	1.5
3a	$B(\frac{1}{2} + i\frac{\sqrt{3}}{2})$; $C(-\frac{1}{2} + i\frac{\sqrt{3}}{2})$; $E(-\frac{1}{2} - i\frac{\sqrt{3}}{2})$ and $F(\frac{1}{2} - i\frac{\sqrt{3}}{2})$	1
3b	The complex form of S is $Z = az + b$ with S(O) = O' ; $O'(\frac{3}{4} + i\frac{\sqrt{3}}{4})$ where $z_{O'} = b$ then $a = \frac{1}{2}e^{i\frac{2\pi}{3}} = -\frac{1}{4} + i\frac{\sqrt{3}}{4}$ and $b = \frac{3}{4} + i\frac{\sqrt{3}}{4}$ therefore $Z = (-\frac{1}{4} + i\frac{\sqrt{3}}{4})z + \frac{3}{4} + i\frac{\sqrt{3}}{4}$; $Z_W = \frac{b}{1-a} = \frac{3}{7} + 2i\frac{\sqrt{3}}{7}$	1
3c	H is the orthocenter of the equilateral triangle OAB, its image has to be the orthocenter of the equilateral triangle O'BJ = S(OAB) ** OR $H(\frac{1}{2} + i\frac{\sqrt{3}}{6})$, its image H' has affix $Z_{H'} = az_H + b = \frac{1}{2} + i\frac{\sqrt{3}}{3}$	0.5

<table>
<tr><th>QVI</th><th>Answer</th><th>Mark</th></tr>
<tr><td>A1</td><td>$f'(x)=2x+\frac{1}{x}$ but $x>0$ then $f'(x)>0$ and for all x in I f is strictly increasing over I.</td><td>0.5</td></tr>
<tr><td>A2a</td><td>$\lim\limits_{x\to0}\ell n\,x=-\infty$ so $\lim\limits_{x\to0^+}f(x)=-\infty$ then $x=0$ (V.A.).</td><td>0.5</td></tr>
<tr><td>A2b</td><td>$\lim\limits_{x\to+\infty}\ell n\,x=+\infty$ then $\lim\limits_{x\to+\infty}f(x)=+\infty$
$\frac{f(x)}{x}=x+\frac{\ell n\,x}{x}$ but $\lim\limits_{x\to+\infty}\frac{\ell n\,x}{x}=0$ hence $\lim\limits_{x\to+\infty}\frac{f(x)}{x}=+\infty$</td><td>1</td></tr>
<tr><td>A2c</td><td>$\begin{array}{c|ccc} x & 0 & & +\infty \\ \hline f'(x) & & + & \\ \hline f(x) & -\infty & \nearrow & +\infty \end{array}$</td><td>0.5</td></tr>
<tr><td>A2d</td><td>f is continuous and strictly increasing over I and it increases from $-\infty$ to $+\infty$, so it vanishes once and changes sign over I. Hence, the equation f(x) = 0 has a unique solution α. Also, $f(0.6)\times f(0.7)<0$ (f(0.6)= - 0.15 and f(0.7)=0.133).
Therefore, $0.6<\alpha<0.7$ and $f(x)<0$ for $0<x<\alpha$; $f(x)>0$ for $x>\alpha$.</td><td>1</td></tr>
<tr><td>A3a</td><td>$f''(x)=2-\frac{1}{x^2}=\frac{2x^2-1}{x^2}$; $f''(x)=0$ for $2x^2=1$ that is for $x=\frac{1}{\sqrt{2}}$ or
$x=\frac{-1}{\sqrt{2}}$ but $x>0$ so $x=\frac{1}{\sqrt{2}}$.
$f''(x)<0$ for $0<x<\frac{1}{\sqrt{2}}$ and $f''(x)>0$ for $x>\frac{1}{\sqrt{2}}$ hence (C) has a point of inflection W of abscissa $\frac{1}{\sqrt{2}}$</td><td>1</td></tr>
<tr><td>A3b</td><td></td><td>0.5</td></tr>
<tr><td>A4a</td><td>f is continuous and strictly increasing over I, then it admits an inverse function f^{-1} defined over $f(I)=\left]-\infty;+\infty\right[$</td><td>0.5</td></tr>
</table>

A4b	f(1) = 1 then A(1;1) is a point common to (C) and (C') since $x_A = y_A$ and $A \in (C)$ so we can draw (C') the symmetric of (C) with respect to the straight line $(\Delta): y = x$.	1
A4c	The tangent at A to (C): $y - y_A = f'(1)(x-1)$ then $y - 1 = 3(x-1)$ and $y = 3x - 2$ hence the tangent at A to (C') has an equation: $x = 3y - 2$ that is $y = \frac{1}{3}x + \frac{2}{3}$	1
A4d	By symmetry with respect to (Δ) we can write: $A(\alpha) = 2\left[\int_0^1 x dx - \int_\alpha^1 f(x) dx\right]$ or $\int_\alpha^1 \ell n\, x dx = \left[x\ell n\, x - x\right]_\alpha^1 = -1 - \alpha\ell n\alpha + \alpha$ $\begin{pmatrix} \text{Let} & u' = 1 & u = x \\ & v = \ell n\, x & v' = \frac{1}{x} \end{pmatrix}$ Then, $A(\alpha) = 2[\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_\alpha^1 + 1 + \alpha\ell n\alpha - \alpha]$ $= 2\left[\frac{1}{2} - \frac{1}{3} + \frac{\alpha^3}{3} + 1 + \alpha\ell n\alpha - \alpha\right]$ $= \frac{7}{3} + \frac{2\alpha^3}{3} + 2\alpha\ell n\alpha - 2\alpha$	1.5
B1	$f(x) - \ln x = x^2 > 0$ for $x > 0$ so (C) is above (T) and draw (T).	1
B2a	$g'(x) = 2x + 2\ln x \times \frac{1}{x} = \frac{2}{x}\left(x^2 + \ln x\right) = \frac{2}{x} f(x)$ so over I, g'(x) has the same sign as f(x).	1
B2b	x: 0 … α … +∞ g'(x): – 0 + g(x): ↘ ↗	0.5
B3a	$OM_0^2 = \alpha^2 + (\ln\alpha)^2 = g(\alpha)$ and $OM^2 = x^2 + (\ln x)^2 = g(x)$	0.5
B3b	$g(\alpha)$ is the minimal value of g(x) hence $g(\alpha) \le g(x)$ for all $x > 0$. Consequently, $OM_0^2 \le OM^2$ That is $OM_0 < OM$ for all points M of (T).	1
B3c	Slope of $(OM_0) = \frac{\ln\alpha}{\alpha}$. Slope of the tangent at M_0 to (T) is $\frac{1}{\alpha}$.Since the derivative of $\ln x$ is $\frac{1}{x}$. But $\alpha^2 + \ln\alpha = 0$ then $\ln\alpha = -\alpha^2$ therefore: $\frac{\ln\alpha}{\alpha} \times \frac{1}{\alpha} = \frac{\ln\alpha}{\alpha^2} = -1$ so the tangent at M_0 to (T) is perpendicular to (OM_0)	1