وزارة التربية والتعليم العالي المديرية العامة للتربية دائرة الامتحانات	امتحانات الشهادة الثانوية العامة الفرع : علوم عامة	دورة سنة 2012 الإستثنائية
عدد المسائل: ست	مسابقة في مادة الرياضيات المدة أربع ساعات	الاسم: الرقم:

ارشادات عامة :- يسمح باستعمال آلة حاسبة غير قابلة للبرمجة او اختزان المعلومات او رسم البيانات
- يستطيع المرشح الإجابة بالترتيب الذي يناسبه (دون الالتزام بترتيب المسائل الوارد في المسابقة)

I- (2points)

In the table below, only one of the proposed answers to each question, is correct.
Write down the number of each question and give, **with justification**, its corresponding answer :

Nº	Questions	Answers			
		a	b	c	d
1	θ is a given real number . If $z = -2e^{-i\theta}$, then an argument of z is :	θ	$\pi - \theta$	$\pi + \theta$	$-\theta$
2	The solution set of the inequality $(\ln x)^2 - 2\ln x < 0$ is :	$[\ 1\ ;\ e^2\ ]$	$]\ e^2 ; +\infty\ [$	$]\ 1\ ;\ e^2\ [$	$]\ 0 ; 2\ [$
3	$\lim\limits_{x \to 0} x \ln\left(1 + \frac{1}{x}\right) =$	0	e	$+\infty$	1
4	If z is a complex number different from i, then $\left\lvert \frac{i\bar{z} - 1}{z - i} \right\rvert =$	$\lvert z \rvert$	1	$\frac{1}{2}$	2

II- (2 points)

In the space referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$, consider the plane (P) with equation $x-y-z+1=0$ and the points A(1 ; 2 ; 0) and B(−1 ; −2 ; 2).

1) Verify that A and B belong to (P) and determine an equation of the plane (Q) passing through A and B and perpendicular to (P).

2) Let (d) be the perpendicular bisector of segment [AB] in (P). Show that a system of parametric equations of (d) is : $\begin{cases} x = t-1 \\ y = 0 \\ z = t \end{cases}$ (t is a real parameter).

3) Consider, in the plane (P), the circle (C) with diameter [AB]. (C) intersects (d) in two points E and F.

a- Calculate the coordinates of the points E and F (E is the point with positive abscissa).

b- Let (T) be the tangent at E to (C) and M any point on (T). Prove that, as M moves on (T), the distance from M to (Q) remains constant.

III- (3 points)

In the plane referred to an orthonormal system $\left(O\,;\vec{i}\,,\vec{j}\right)$, consider the curve (C_m) with equation:

$mx^2 - 2y^2 + 2mx + 4y = 0$ with m being a real number different from −2 , 0 and 2.

A- In this part, take m =1.

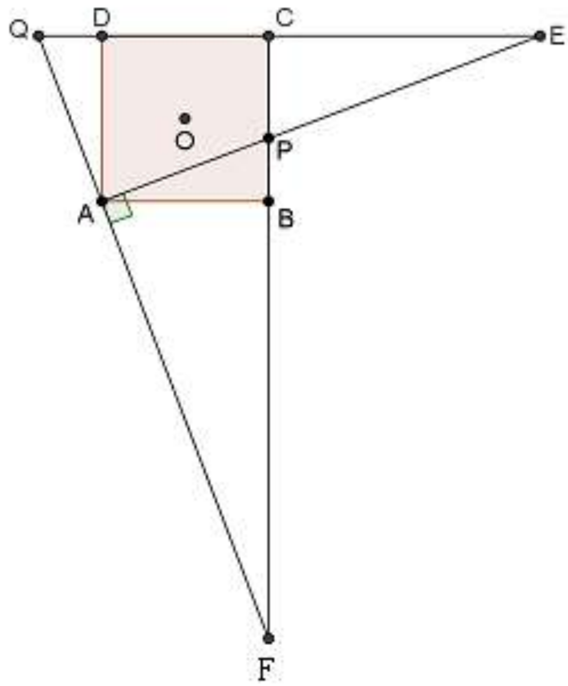
1) Prove that (C_1) is a hyperbola whose center I and focal axis are to be determined.
2) Calculate the coordinates of the vertices of (C_1) and determine its asymptotes.
3) Draw (C_1).
4) The tangent and the normal to (C_1) at O intersect the line with equation x= −1 in two points T and N respectively. Prove that $\overrightarrow{IT}.\overrightarrow{IN} = \frac{3}{2}$.

B - In this part, suppose that m < 0.

1) Verify that $\dfrac{(x+1)^2}{\frac{m-2}{m}} + \dfrac{(y-1)^2}{\frac{2-m}{2}} = 1$ is an equation of (C_m). Deduce that (C_m) is an ellipse.

2) Determine the set of values of m so that the focal axis of (C_m) is parallel to the x-axis.

IV- (3 points)



In the figure to the right, ABCD is a square with side 1 and center O such that $\left(\overrightarrow{AB},\overrightarrow{AD}\right)=\frac{\pi}{2}(2\pi)$.

P is a point on the segment [BC] such that PB = t with 0 < t < 1.
The line (AP) intersects the line (CD) at E.
The perpendicular to (AP) at A intersects (CB) at F and (CD) at Q.
Denote by M the midpoint of [FE] and by N that of [PQ].

1) Let r be the rotation with center A and angle $\frac{\pi}{2}$.

a- Determine, with justification, the image of (BC) under r.
b- Show that r(P)= Q and determine r(F) .
c- Specify the nature of each of the triangles APQ and AFE.

2) Let s be the similitude with center A, ratio $\frac{1}{\sqrt{2}}$ and angle $\frac{\pi}{4}$.

a- Prove that s(P) = N; and determine s(F) and s(B).
b- Deduce that M, B, N and D are collinear.

3) a- Prove that $BF=\frac{1}{t}$.

b- Determine t so that the area of triangle AMN is equal to $\frac{5}{8}$.

4) The complex plane is referred to the system $\left(A;\overrightarrow{AB},\overrightarrow{AD}\right)$.

a- Write the complex form of s.

b- In the case where $t=\frac{1}{3}$, determine the affixes z_M and z_N of the points M and N and deduce that $\frac{z_M-1}{z_N-1}$ is a real number.

V- (3 points)

An urn contains **5** red balls, **4** black balls and **3** green balls. Three balls are randomly selected from the urn. Consider the following events:

E: « The three selected balls have the same color»
F: « The three selected balls have three different colors»
G: « Only two of the three selected balls have the same color» .

A- In this part, the selection of the three balls is done **simultaneously**.

1) Calculate the probabilities p(E) , p(F) and p(G) .
2) Knowing that only two of the three selected balls have the same color, calculate the probability that the third ball is red .

B- In this part, the selection of the three balls is done successively and **with replacement**.
1) Calculate p(E) and p(F) . Deduce p(G) .
2) Let X be the random variable equal to the number of red balls among the three selected balls.

a- Prove that $p(X=2)=\frac{175}{576}$.

b- Determine the probability distribution of X.

VI - (7 points)

Consider the two functions f and g defined over $]0; +\infty[$ by:

$f(x) = 2x + \frac{1 - \ln x}{x}$ and $g(x) = 2x^2 - 2 + \ln x$.

Denote by (C) the representative curve of f in an orthonormal system $(O; \vec{i}, \vec{j})$.

A- 1) Show that g is strictly increasing over $]0; +\infty[$.

2) Calculate g(1) and deduce the sign of g(x) according to the values of x.

B- 1) a- Determine $\lim\limits_{x \to 0} f(x)$ and deduce an asymptote to (C).

b- Show that the line (d) with equation y = 2x is an asymptote to (C). Study, according to the values of x, the relative positions of (C) and (d).

2) Show that $f'(x) = \frac{g(x)}{x^2}$.

3) Set up the table of variations of f.

4) Draw (d) and (C) in the system $(O; \vec{i}, \vec{j})$.

5) a- Show that f has over $[1; +\infty[$ an inverse function h whose domain of definition is to be determined.

b- Draw (Γ), the representative curve of h in the same system as that of (C).

c- Determine the abscissa of the point of (Γ) where the tangent is parallel to the line with equation $y = \frac{x}{2}$.

6) a- For all natural numbers n, let $U_n = \int_{e^n}^{e^{n+1}} [f(x) - 2x]\,dx$.

Calculate U_n and prove that (U_n) is an arithmetic sequence whose common difference is to be determined.

b- Let A be the area, in square units, of the region bounded by (C), (d) and the two lines with equations x = 1 and $x = e^2$. Verify that A is equal to $U_o - U_1$.

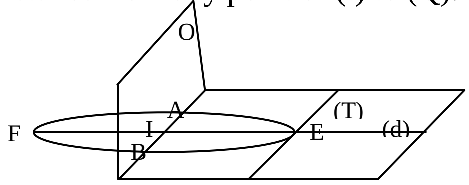
C- Consider the function p defined over $]0; +\infty[$ by: $p(x) = x^2(1 + \ln x) - 3x + 2$.

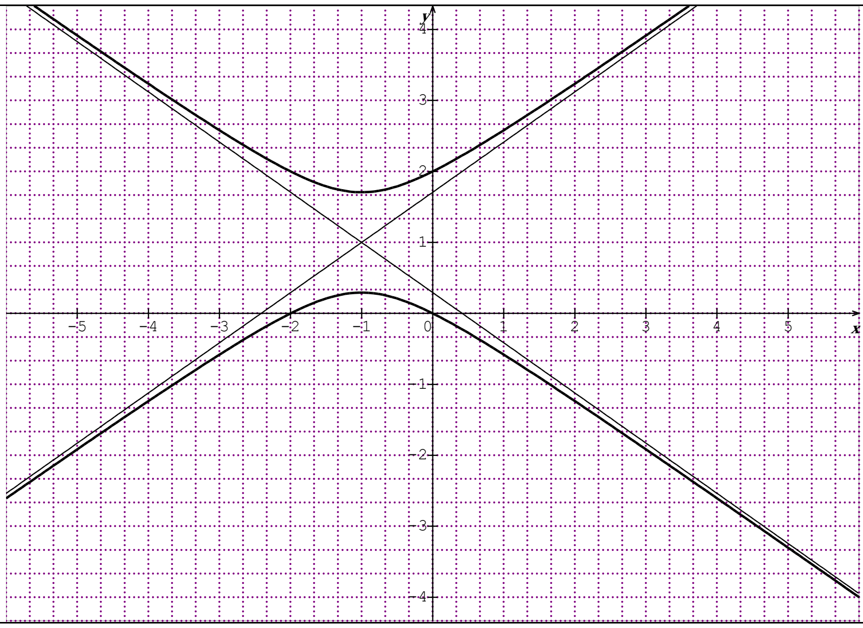
1) Show that p can be extended by continuity at the point x = 0.

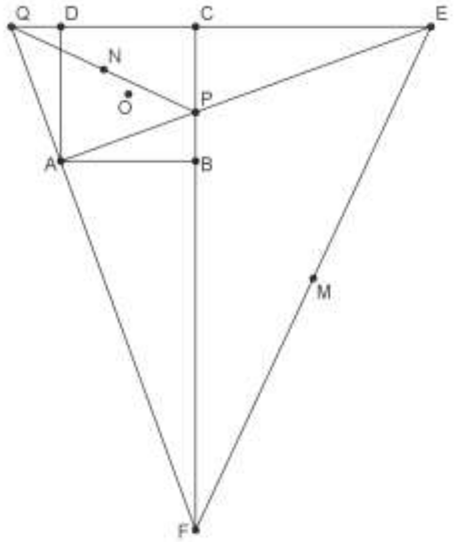
2) For all real numbers x in $]0; +\infty[$, prove that $\frac{p(x)}{x} = f\left(\frac{1}{x}\right) - 3$ and that $p(x) \geq 0$.

3) Specify the value of x so that p(x) = 0.

I	Answers		Grade
1	$z = -2\,e^{-i\theta} = 2\,e^{-i\theta+\pi} = 2\,e^{i(\pi-\theta)}$.	b	1
2	$(\ln x)^2 - 2\ln x < 0$; $\ln x(\ln x-2)<0$ thus $0<\ln x<2$ so $1<x<e^2$	b	1
3	$\lim\limits_{x\to 0} x\ln(1+\frac{1}{x}) = \lim\limits_{t\to+\infty}\frac{\ln(1+t)}{t} = 0$, where $t = \frac{1}{x}$.	a	1
4	$\frac{\lvert i\bar{z}+1\rvert}{\lvert z-i\rvert} = \frac{\lvert i(\bar{z}-i)\rvert}{\lvert z+i\rvert} = \frac{\lvert i\rvert\lvert\overline{z-i}\rvert}{\lvert z-i\rvert} = \frac{\lvert z-i\rvert}{\lvert z-i\rvert}$ 1 .	d	1

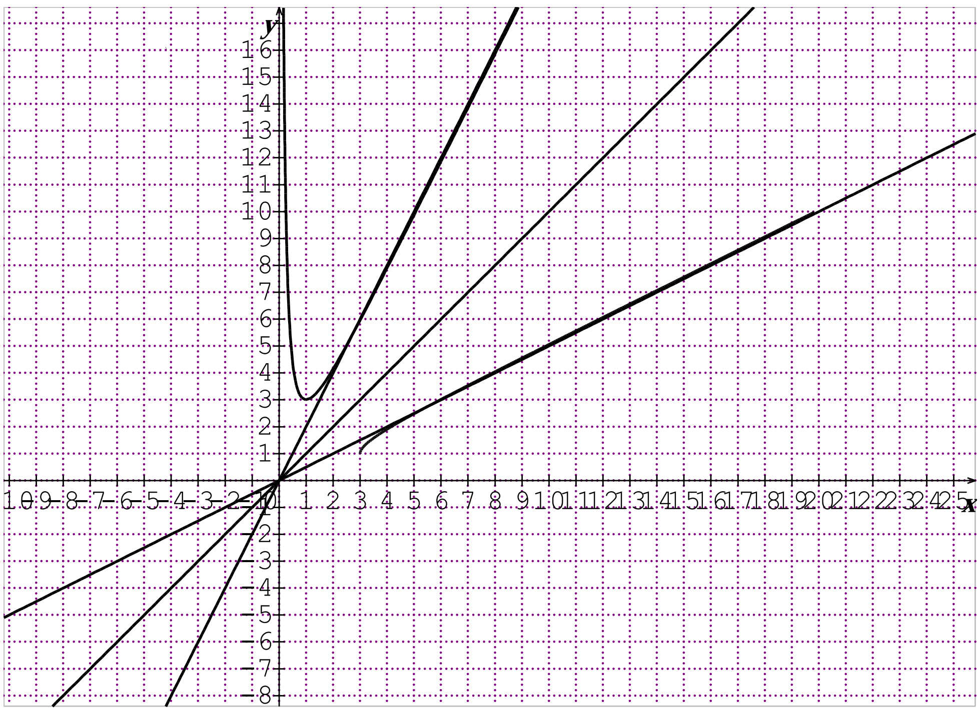
II	Answers	Grade
1	$1-2-0+1=0$ then $A\in(P)$ and $-1+2-2+1=0$ so $B\in(P)$. (Q) is perpendicular to (P) so $\overrightarrow{n_P}$ is parallel to (Q). If $M(x ; y ; z)$ is a point of (Q), then $\overrightarrow{AM}\cdot\left(\overrightarrow{n_P}\wedge\overrightarrow{AB}\right)=0$; and $x+z-1=0$	1
2	(d) is the intersection of (P) with the mediator plane (R) of [AB], an equation of (R) is: $\overrightarrow{IM}\cdot\overrightarrow{AB}=0 \Leftrightarrow x(-2)+y(-4)+(z-1).2=0$. $x+2y-z+1=0$. Where I is the midpoint of [AB] and $M(x;y;z)\in(R)$. Let $z=t$ then $\begin{cases}x-y=t-1 & \text{so} & y=0\\ x+2y=t-1 & \text{and} & x=t-1\end{cases}$ Hence $(d):\begin{cases}x=t-1\\ y=0\\ z=t\end{cases}$ OR: we prove that (d) lies in (P), is perpendicular to [AB] and passes through I , the midpoint of [AB] .	1
3a	E and $F\in(d)$ then $x=t-1$; $y=0$ and $z=t$. E and $F\in(C)$ then $IE=IF=\frac{AB}{2}=\sqrt{6}$ but $I(0;0;1)$ therefore $(t-1)^2+0+(t-1)^2=6$ then $t-1=\sqrt{3}$ or $t-1=-\sqrt{3}$ Hence, $E\left(\sqrt{3};0;1+\sqrt{3}\right)$ and $F\left(-\sqrt{3};0;1-\sqrt{3}\right)$.	1
3b	$(T)\perp(d)$ then (T) is parallel to (AB) then (T) is parallel to (Q) since $(AB)\subset(Q)$. (T) is parallel to (Q) then for all points M of (T) the distance of M to (Q) is the same, it is equal to the radius of (C) that is $\sqrt{6}$. OR determine a parametric representation of (T) (a directing vector of (T) is $\overrightarrow{AB}$) and we calculate the distance from any point of (t) to (Q). 	1

III	Answers	Grade
A1	For m = 1 ; $x^2-2y^2+2x+4y=0$ thus $(x+1)^2-2(y-1)^2=-1$ $\frac{(y-1)^2}{\frac{1}{2}}-(x+1)^2=1$ then (C_1) is a hyperbola with center I(–1 ; 1) and focal axis passing through I and parallel to the y-axis so it has an equation : x = –1.	1
A2	For x = –1 $(y-1)^2=\frac{1}{2}\to y=1+\frac{1}{\sqrt{2}}$ or $y=1-\frac{1}{\sqrt{2}}$ .So, the vertices of (C_1) are $A(-1;1+\frac{1}{\sqrt{2}})$ and $A'\ (-1;1-\frac{1}{\sqrt{2}})$.	1
A3		0.5
A4	Deriving both sides : 2x – 4yy'+2 + 4y' = 0. Thus, $y'=\frac{x+1}{2(2y-1)}$. The slope of the tangent at O is $-\frac{1}{2}$ and that of the normal is 2. The tangent has an equation $y=-\frac{1}{2}x$ and the normal has an equation y = 2x. T(–1 ; $\frac{1}{2}$) and N(–1 ;–2). $\overrightarrow{IT}\cdot\overrightarrow{IN}=\frac{3}{2}$.	1.5
B1	$m(x+1)^2-2(y-1)^2=m-2$. $\frac{m(x+1)^2}{m-2}-\frac{2(y-1)^2}{m-2}=1\leftrightarrow\frac{(x+1)^2}{\frac{m-2}{m}}+\frac{(y-1)^2}{\frac{2-m}{2}}=1$ For m < 0, $\frac{m-2}{m}>0$ and $\frac{2-m}{2}>0$ then (C_m) is an ellipse of center I.	1
B2	The focal axis is $(I;\vec{i})$ when $a^2>b^2$ that is $\frac{m-2}{m}-\frac{2-m}{2}>0\leftrightarrow(m-2)\left(\frac{1}{m}+\frac{1}{2}\right)>0\leftrightarrow\frac{m+2}{2m}<0\leftrightarrow m+2>0$ which gives –2 < m < 0.	1

IV	Answers	Grade
1a	$r(B) = D$ then the image of (BC) under r is the line passing through D and perpendicular to (BC). Hence r (BC) = (DC). 	0.5
1b	$P \in (BC)$ then $r(P) \in (DC)$ and $r(P)$ belongs to the line passing through A and perpendicular to (AP) which is (AQ) thus r(P) = Q. Similarly, r (F) = (DC) $\cap$ (AP) = E .	0.5
1c	$AP = AQ$ and $\left(\overrightarrow{AP}, \overrightarrow{AQ}\right) = \frac{\pi}{2}[2\pi]$ thus APQ is right isosceles. $AF = AE$ and $\left(\overrightarrow{AF}, \overrightarrow{AE}\right) = \frac{\pi}{2}[2\pi]$ thus AFE is right isosceles.	0.5
2a	$s(P) = N$ since $\left(\overrightarrow{AP}, \overrightarrow{AN}\right) = \frac{\pi}{4}[2\pi]$ and $\frac{AN}{AP} = \frac{1}{\sqrt{2}}$ triangle ANP being right isosceles. $s(F) = M$ since $\left(\overrightarrow{AF}, \overrightarrow{AM}\right) = \frac{\pi}{4}[2\pi]$ and $\frac{AM}{AF} = \frac{1}{\sqrt{2}}$ triangle AMF being right isosceles. s(B) = O since ABO is right isosceles with vertex O.	1
2b	$s(P) = N$; $s(F) = M$; $s(B) = O$ and $s(C) = D$ but P, F, B and C are collinear and the image under s of a line is a line . hence N,M,O and D are collinear and B belongs to (OD).	1
3a	$(\widehat{APB}) = (\widehat{BAF})$ but $\tan(\widehat{APB}) = \frac{AB}{BP} = \frac{1}{t}$ and $\tan(\widehat{BAF}) = \frac{BF}{AB} = BF$, therefore $BF = \frac{1}{t}$. OR: In the right triangle APF : $AB^2 = BP \times BF$.	0.5
3b	$s(AFP) = AMN$ the $\text{Area}(AMN) = \left(\frac{1}{\sqrt{2}}\right)^2 \times \text{Area } AFP$ But $\text{Area}(AFP) = \frac{AB \times PF}{2} = \frac{1 \times \left(t + \frac{1}{t}\right)}{2} = \frac{t^2+1}{2t}$ Therefore $\frac{5}{8} = \frac{1}{2} \times \frac{t^2+1}{2t} \Rightarrow 2t^2 - 5t + 2 = 0$ and $0 < t < 1$ $t = \frac{1}{2}$	0.5

4a	$z'=\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}z$ since A is the origin. $z'=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)z=\frac{1+i}{2}z$.	0.5
4b	$z_P=1+\frac{i}{3}$ and $z_F=1-3i$ $s(P)=N \Leftrightarrow z_N=\frac{(1+i)}{2}\left(1+\frac{i}{3}\right)=\frac{1+2i}{3}$ $s(F)=M \Leftrightarrow z_M=\left(\frac{1+i}{2}\right)(1-3i)=2-i$ $\frac{z_M-1}{z_N-1}=\frac{\frac{1+2i}{3}-1}{2-i-1}=\frac{-2+2i}{3(1-i)}=-\frac{2}{3}$. Thus, it is a real number.	1

V	**Answers**	**Grade**
A1	p(E) = $\frac{C_5^3}{C_{12}^3}+\frac{C_4^3}{C_{12}^3}+\frac{C_3^3}{C_{12}^3}=\frac{3}{44}$. p(F) = $\frac{C_5^1\times C_4^1\times C_3^1}{C_{12}^3}=\frac{3}{11}$. p(G) = 1 – [p(E) + p(F)] = $\frac{29}{44}$. ***or*** p(G) = $\frac{C_5^1\times C_7^1+C_4^2\times C_5^1+C_3^2\times C_5^1}{C_{12}^3}=\frac{29}{44}$.	1.5
A2	$p=\frac{p(\text{2 black and 1 red})+p(\text{2 green and 1 red})}{p(G)} = \frac{C_5^1\times C_4^2+C_5^1\times C_3^2}{C_{12}^3}\div\frac{29}{44}=\frac{9}{29}$.	1
B1	p(E) = $(\frac{5}{12})^3+(\frac{4}{12})^3+(\frac{3}{12})^3=\frac{1}{8}$. p(F) = $(\frac{5}{12}\times\frac{4}{12}\times\frac{3}{12})\times 3!=\frac{5}{24}$. p(G) = 1 – (p(E) + p(F)) = $\frac{2}{3}$.	1.5
B2a	2 red balls are selected from the urn ; then $p(X=2)=\left(\frac{5}{12}\right)^2\times\frac{7}{12}\times 3=\frac{175}{576}$	0.5
B2b	The possible values of X are : 0 , 1 , 2 , 3 . p(X=0)= $\left(\frac{7}{12}\right)^3=\frac{343}{1728}$ p(X = 1) = $\frac{5}{12}\times(\frac{7}{12})^2\times 3=\frac{245}{576}$ p(X = 2) = $\frac{175}{576}$ p(X = 3) = $(\frac{5}{12})^3$.	1.5

VI	Answer	Grade
A1	g’(x) = 4x + (1/x) > 0 ; g is strictly increasing.	0.5
A2	g(1) = 0 then g(x) = 0 for x = 1 , g(x) < 0 for 0< x < 1 and g(x) > 0 for x > 1.	0.5
B1a	$\lim\limits_{x\to 0} f(x) = 0 + \frac{+\infty}{0^+} = +\infty$; the y-axis is an asymptote to (C).	0.5
B1b	$\lim\limits_{x\to +\infty}\left[f(x) - 2x\right] = \lim\limits_{x\to +\infty}\left[\frac{1}{x} - \frac{\ln x}{x}\right] = 0$; y = 2x is an equation of an asymptote to (C) as x tends to $+\infty$. $f(x)-2x=\frac{1-\ln x}{x}$. For x<e, f(x)-2x>0 so (C) is above (d). For x=e, f(x)-2x=0 so (C) and (d) intersect at the point I(e,2e) For x>e, f(x)-2x< 0 so (C) is below (d).	1
B2	$f'(x) = 2 + \frac{-1-1+\ln x}{x^2} = \frac{g(x)}{x^2}$.	0.5
B3	$\begin{array}{c\|ccccc} x & 0 & & 1 & & +\infty \\ \hline f'(x) & \| & - & 0 & + & \\ \hline f(x) & \| +\infty & \searrow & 3 & \nearrow & +\infty \end{array}$	0.5
B4		1.5
B5a	f is continuous and strictly increasing over $[1;+\infty[$ then it has an inverse function h defined over $[3;+\infty[$	0.5
B5b	(Γ) is the symmetric of (C) with respect the straight line of equation $y = x$, refer to the figure.	1.5
B5c	We can find the point on (C) where the tangent is parallel to the line $y = 2x$. $f'(x) = 2$ $g(x) = 2x^2$; $\ln x = 2$; $x = e^2$, the required point on (C) is (e^2 ; $2e^2 - e^{-2}$). The required point on (Γ) is ($2e^2 - e^{-2}$;e^2)	1

B6a	$U_n = \int_{e^n}^{e^{n+1}} \frac{1-\ln x}{x}\,dx = \left[-\frac{(1-\ln x)^2}{2}\right]_{e^n}^{e^{n+1}} = \frac{1}{2}[(1-n)^2 - n^2] = -n + \frac{1}{2}$ $U_{n+1} - U_n = -1$, (U_n) is an arithmetic sequence of common difference d = –1.	2
B6b	$A = \int_1^e \frac{1-\ln x}{x}\,dx - \int_e^{e^2} \frac{1-\ln x}{x}\,dx = U_0 - U_1 = 1$ square unit	1.5
C1	$\lim\limits_{\substack{x\to 0\\ x>0}} p(x) = \lim\limits_{\substack{x\to 0\\ x>0}} x(x + x\ln x - 3) + 2 = 2$ Then p can be extended by continuity at the point O.	1
C2	$\frac{p(x)}{x} = x(1+\ln x) - 3 + \frac{2}{x} = f(\frac{1}{x}) - 3 \geq 0$ (f has a minimum equal to 3) then $p(x) \geq 0$.	1
C3	p(x)=0 for $f(\frac{1}{x})=3$ then $\frac{1}{x}=1$ so x=1	0.5