وزارة التربية والتعليم العالي المديرية العامة للتربية دائرة الامتحانات	امتحانات شهادة الثانوية العامة فرع العلوم العامة	دورة العام 2013 الاستثنائية الإثنين 19 آب 2013
عدد المسائل ست	مسابقة في الرياضيات المدة أربع ساعات	الإسم: الرقم:

ملاحظة: يسمح باستعمال آلة حاسبة غير قابلة للبرمجة أو تخزين المعلومات أو رسم البيانات
يستطيع المرشح الاجابة بالترتيب الذي يناسبه (دون الالتزام بترتيب المسائل الوارد في المسابقة)

I- (2 points)

In the space referred to an orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$, consider the three non-collinear points A (1 ; 1 ; 1), B (– 2 ; – 5 ; – 2) and I (4 ; – 2 ; 4).
In the plane (ABI), consider the circle (C) with center I and radius R = $3\sqrt{3}$.

1) Verify that A is a point on (C) and B is outside (C).

2) Show that (AB) is tangent to (C).

3) Let (BT) be the second tangent through B to (C). (T is on (C))
Calculate the area of the quadrilateral AITB.

4) Let (Q) be the plane passing through A and perpendicular to the line (BI).

   a- Determine an equation of (Q).
   b- Write a system of parametric equations of (BI).
   c- Let H be the point of intersection of (Q) and the line (BI).
   Calculate the coordinates of H and deduce the coordinates of T.

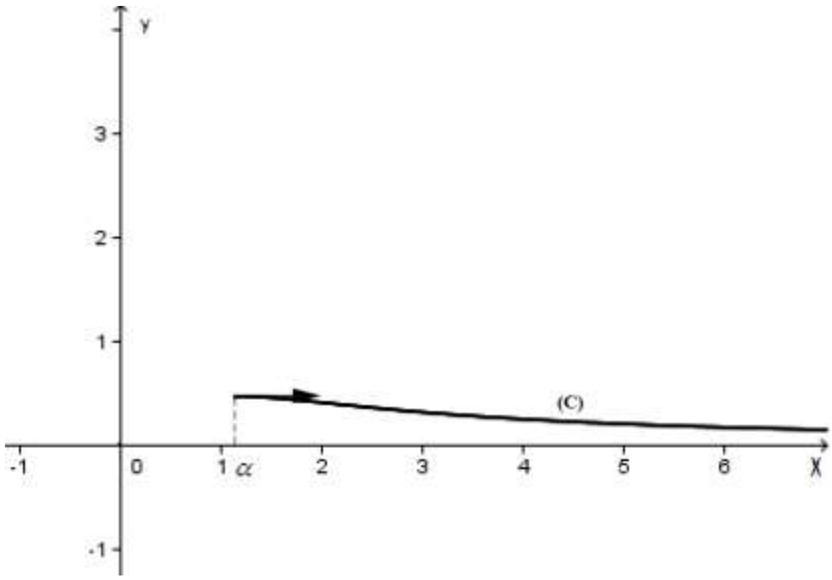
II- (2 points)

Let α be a real number greater than 1.

The adjacent curve (C) represents on $[\alpha\,;+\infty[$ the function f defined as

$f(x) = \dfrac{1-e^{-x}}{x+e^{-x}}$.

The x-axis is an asymptote to (C).



1) Determine an antiderivative F of f on $[\alpha;+\infty[$.

2) For every natural number $n>1$, let $u_n = \int_n^{n+1} f(x)\,dx$.

a- Calculate u_2 and u_3 to the nearest 10^{-2}.

b- For $n \le x \le n+1$, show that $f(n+1) \le f(x) \le f(n)$ and deduce that $f(n+1) \le u_n \le f(n)$.

c- Deduce that the sequence (u_n) is decreasing.

d- Determine the limit of the sequence (u_n).

III- (3 points)

The complex plane is referred to a direct orthonormal system $(O\,;\vec{u},\vec{v})$.

To every point M(x ; y) with affix $z \neq 0$, associate the point M´ (x′ ; y′) with affix z´ so that $z' = z + \frac{1}{z}$.

1) Prove that $x' = x\left(1+\frac{1}{x^2+y^2}\right)$ and $y' = y\left(1-\frac{1}{x^2+y^2}\right)$.

2) Assume that M moves on the circle with equation $x^2+y^2=4$.
Express x' and y' in terms of x and y, then prove that M' moves on the ellipse (E) with equation $36x^2+100y^2=225$.

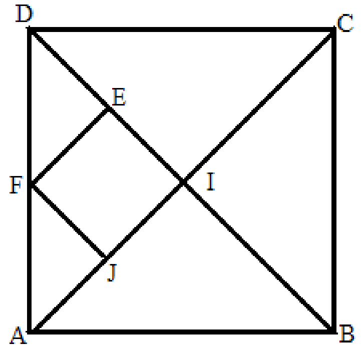
3) Assume that M moves on the line with equation y = x, deprived from O.
Verify that M´ moves on the hyperbola (H) with equation $x^2 - y^2 = 2$.

4) Prove that the ellipse (E) and the hyperbola (H) have the same foci.

5) Let $A\left(\frac{5\sqrt{2}}{4},\frac{3\sqrt{2}}{4}\right)$ be one of the common points of (E) and (H).

Prove that the tangents at A to (E) and (H) are perpendicular.

IV- (3 points)



ABCD is a direct square with center I.

Denote by E, F and J the respective midpoints of the segments [ID], [DA] and [AI].

Let S be the similitude with ratio k and an angle α that maps A onto J and B onto I.

1) Calculate k and determine a measure of α.

2) Prove that E is the image of C under S and determine S(D), S(I) and S(J).

3) a- Specify the nature of $S \circ S$, then calculate its ratio and a measure of its angle.
   b- Determine SoS(A) and SoS(B), then construct the point Ω, center of S.

4) The plane is referred to a direct orthonormal system $(A\,;\overrightarrow{AB},\overrightarrow{AD})$.
   Determine the complex form of S, then deduce the coordinates of the point Ω.

V- (3 points)

Consider two dice so that one of them is a fair die and the other one is biased.

The faces of every die are numbered from 1 to 6.

If the fair die is rolled, all the faces have the same probability to appear.

If the biased die is rolled, the probability to obtain the face carrying the number 4 is equal to $\frac{1}{3}$, while all the other faces have the same probability to appear .

1) The biased die is rolled once.

   Prove that the probability of obtaining a face not carrying the number 4 is equal to $\frac{2}{15}$.

2) The fair die is rolled twice.

   Let X be the random variable equal to the number of times when the face carrying the number 4 appears.

   Determine the probability distribution of X.

3) The biased die is rolled twice.

   Show that the probability to obtain the face carrying the number 4 exactly one time is equal to $\frac{4}{9}$.

4) One of the two dice is chosen randomly, and then it is rolled twice.

   (The two dice have the same probability to be chosen).

   Calculate the probability to obtain exactly one face carrying the number 4.

VI- (7 points)

Let f be the function defined, over [0 ; + ∞[, as $\begin{cases} f(x) = x(\ln x - 1)^2 \text{ for } x > 0 \\ f(0) = 0 \end{cases}$

and let (C) be its representative curve in an orthonormal system $(O ; \vec{i}, \vec{j})$.

We admit that f is continuous at x = 0.

1) a- Determine $\lim\limits_{x \to 0} \frac{f(x)}{x}$. Give a graphical interpretation to the result.

   b- Find $\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to +\infty} \frac{f(x)}{x}$.

2) For $x > 0$, verify that $f'(x) = (\ln x)^2 - 1$ and set up the table of variations of f.

3) Show that (C) has a point of inflection I and write an equation of (T), the tangent at I to (C).

4) The line (Δ) with equation y = x intersects (C) at three points O, I and J.
   Calculate the coordinates of J.

5) Plot (T) and (C).

6) a- Show that the function F defined on]0 ; +∞[as $F(x) = \frac{x^2}{2}\left[(\ln x)^2 - 3\ln x + \frac{5}{2}\right]$
   is an antiderivative of f.

   b- Deduce the area of the region bounded by the x-axis, the tangent (T) and the curve (C).

7) For all x in the interval [e ; + ∞[, prove that f has an inverse function f^{-1} and plot the representative curve of f^{-1} in the same system as (C).

8) Let (d_m) be the line with equation $y = mx$ where $m > 0$.
   The line (d_m) intersects the curve (C) at three distinct points O, M and M´.
   a- Calculate, in terms of m, the coordinates of the points M and M´.
   b- Denote by P the point on (d_m) with abscissa x = e.
   Prove that $\overrightarrow{OM}.\overrightarrow{OM'} = OP^2$.

Mathematics	Bareme GS	Second Session 2013

Q_1	Answers	Mark
1	$\overrightarrow{AI}(3;-3;3)$ IA= $3\sqrt{3}$ =R, so A is on the circle. $\overrightarrow{BI}(6;3;6)$ IB = 9 > R, then B is outside (C).	0.5
2	(AB) is in (ABI). $\overrightarrow{AB}(-3;-6;-3)$, $\overrightarrow{AI}(3;-3;3)$ thus $\overrightarrow{AB}\cdot\overrightarrow{AI}=0$, and (AB) is tangent to (C).	0.5
3	The area of AITB = 2× area of triangle AIB = AI× AB = $27\sqrt{2}$ units of area.	0.5
4a	$\vec{n}_Q$ is parallel to vector $\overrightarrow{BI}(6;3;6)$, thus $\vec{n}_Q(2;1;2)$, then an equation of (Q) is : 2x + y + 2z + r = 0, and since $A\in(Q)$, then r = –5, and (Q) : 2x + y + 2z –5 = 0.	0.5
4b	$\overrightarrow{BI}(6;3;6)$, therefore $\vec{v}_{BI}(2;1;2)$ and (BI) passes through I then (BI) : $\begin{cases} x = 2m+4 \\ y = m-2 \\ z = 2m+4 \end{cases}$ where $m\in\mathbb{R}$.	0.5
4c	The point H is on (BI), so $x_H = 2m+4$, $y_H = m-2$ and $z_H = 2m+4$. Also, H $\in$(Q) so $2x_H + y_H + 2z_H - 5 = 0$ so $m=-1$; Thus H (2 ;-3 ;2). T is the symmetric of A with respect to H, therefore H is midpoint of [AT], so $T(3;-7;3)$.	1.5

Q_2	Answers	Mark
1	$F(x) = \int f(x)\,dx = \int \frac{\left(x+e^{-x}\right)'}{\left(x+e^{-x}\right)}dx = \ln\left(x+e^{-x}\right)+c.$	1
2.a	$u_2 = \int_2^3 f(x)\,dx = F(3)-F(2) = \ln\left(3+e^{-3}\right)-\ln\left(2+e^{-2}\right) \simeq 0.36.$ $u_3 = \int_3^4 f(x)\,dx = F(4)-F(3) = \ln\left(4+e^{-4}\right)-\ln\left(3+e^{-3}\right) = 1.39-1.12 = 0.27$	0.5
2.b	f is decreasing; for $n \le x \le n+1$, we have $f(n+1) \le f(x) \le f(n)$ $\Rightarrow \int_n^{n+1} f(n+1)\,dx \le \int_n^{n+1} f(x)\,dx \le \int_n^{n+1} f(n)\,dx \Rightarrow f(n+1)\times x\Big]_n^{n+1} \le u_n \le f(n)\times x\Big]_n^{n+1}$ $\Rightarrow f(n+1) \le u_n \le f(n).$	1
2.c	$f\left(n+2\right) \le u_{n+1} \le f(n+1)$; thus $u_{n+1} \le f(n+1) \le u_n$, so (u_n) is decreasing.	1
2.d	$f(n+1) \le u_n \le f(n)$; but $\lim\limits_{n\to\infty} f(n+1) = \lim\limits_{n\to\infty} f(n) = 0$ (Curve), $\lim\limits_{n\to\infty} u_n = 0.$(Sandwich theorem)	0.5

Q_3	Answers	Mark
1	$x'+iy'=x+iy+\frac{1}{x+iy}=x+iy+\frac{x-iy}{x^2+y^2}=x+\frac{x}{x^2+y^2}+i\left(y-\frac{y}{x^2+y^2}\right)$ $x'=x\left(1+\frac{1}{x^2+y^2}\right)$ and $y'=y\left(1-\frac{1}{x^2+y^2}\right)$.	1
2	$x^2+y^2=4$; then $x'=\frac{5x}{4}$ and $y'=\frac{3y}{4}$. By replacing we get : $36\left(\frac{25x^2}{16}\right)+100\left(\frac{9y^2}{16}\right)=\frac{900}{16}\left(x^2+y^2\right)=\frac{900\times 4}{16}=225$. Therefore M' moves on the ellipse with equation $36x^2+100y^2=225$.	1.5
3	Since $x=y$ then $x'=x\left(1+\frac{1}{2x^2}\right)$ and $y'=x\left(1-\frac{1}{2x^2}\right)$ $x'^2-y'^2=x^2\left(1+\frac{1}{2x^2}\right)^2-x^2\left(1-\frac{1}{2x^2}\right)^2=2$, thus $M'\in(H)$.	1.5
4	(E) : $\frac{x^2}{\frac{25}{4}}+\frac{y^2}{\frac{9}{4}}=1$. Hence its center is O its focal axis is $x'Ox$, $c^2=a^2-b^2=\frac{25}{4}-\frac{9}{4}=4$ and c = 2. (H) has center O and focal axis $x'Ox$, $c^2=a^2+b^2=2+2=4$ and c = 2. Hence (E) and (H) have the same foci.	0.5
5	$72x+200yy'=0$ then $y'=-\frac{9x}{25y}$; $2x-2yy'=0$ so $y'=\frac{x}{y}$. Product of slopes $=-\frac{9x^2}{25y^2}=-\frac{9\times 50}{25\times 18}=-1$. the tangents are perpendicular.	1.5

Q_4	Answers	Mark
1	$k=\frac{JI}{AB}=\frac{\frac{1}{4}AC}{AB}=\frac{\frac{1}{4}AB\sqrt{2}}{AB}=\frac{\sqrt{2}}{4}$. $\alpha=\left(\overrightarrow{AB};\overrightarrow{JI}\right)=\frac{\pi}{4}$.	0.5
2	$\frac{IE}{BC}=\frac{\frac{1}{4}BD}{BC}=\frac{\sqrt{2}}{4}$ and $\left(\overrightarrow{BC};\overrightarrow{IE}\right)=\left(\overrightarrow{BC};\overrightarrow{BD}\right)=\frac{\pi}{4}(2\pi)$ thus S(C) = E. **OR :** The triangle ABC is direct right isosceles at B, therefore triangle JIC' should be direct right isosceles at I. Since JIE direct right isosceles at I thus S(C) = E ; ABCD direct square therefore JIED' is also direct square, thus S(D) = F since EF = IJ, FJ = EI = IJ and (IE) is perpendicular to (IJ); I is the center of square ABCD therefore S(I) =I' is the center of square JIEF. J is the midpoint the [AI], so S(J) = J′ is the midpoint of [JI′].	2
3.a	SoS is a similitude with ratio $\frac{\sqrt{2}}{4}.\frac{\sqrt{2}}{4}=\frac{1}{8}$ and angle $\frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$.	1
3.b	SoS(A) = S(J) = J′, SoS(B) = S(I) = I′,	1.5

	$\left(\overrightarrow{\Omega A};\overrightarrow{\Omega J'}\right)=\frac{\pi}{2}$ and $\left(\overrightarrow{\Omega B};\overrightarrow{\Omega I'}\right)=\frac{\pi}{2}$, so Ω is on the circle diameter [AJ′] and Ω is on the circle diamètre [BI′]. Ω is one of the two points of intersection of these two circles where Ω is such that $\left(\overrightarrow{\Omega A};\overrightarrow{\Omega J}\right)=\frac{\pi}{4}$.	
4	$z_C = 1+i$ et $z_I=\frac{z_A+z_C}{2}=\frac{1}{2}+\frac{i}{2}$. S has a complex form $z' = az + b$ where $a=\frac{\sqrt{2}}{4}e^{i.\frac{\pi}{4}}=\left(\frac{1}{4}+\frac{i}{4}\right)$, so $z'=\left(\frac{1}{4}+\frac{i}{4}\right)z+b$. Since S(B) = I $\Rightarrow z_I=\left(\frac{1}{4}+\frac{i}{4}\right)z_B+b\Rightarrow b=\frac{1}{4}+\frac{i}{4}$ and $z'=\left(\frac{1}{4}+\frac{i}{4}\right)z+\frac{1}{4}+\frac{1}{4}i$. $z_\omega=\frac{b}{1-a}=\frac{1}{5}+\frac{2i}{5}\Rightarrow\Omega\left(\frac{1}{5};\frac{2}{5}\right)$.	1

Q_5	Answers	Mark
1	$P(4)=\frac{1}{3}$ and $P(1)=P(2)=P(3)=P(5)=P(6)=\frac{1}{5}\left(1-\frac{1}{3}\right)=\frac{2}{15}$.	1
2.a	$X(\Omega)=\{0;1;2\}$ $P(X=0)=P\left(\overline{4},\overline{4}\right)=\frac{5}{6}\times\frac{5}{6}=\frac{25}{36}$. $P(X=1)=P\left(4,\overline{4}\right)+P\left(\overline{4},4\right)=2\times\frac{5}{6}\times\frac{1}{6}=\frac{5}{18}$. $P(X=2)=P\left(4,4\right)=\frac{1}{6}\times\frac{1}{6}=\frac{1}{36}$.	2
2.b	$P(4)=P\left(4,\overline{4}\right)+P\left(\overline{4},4\right)=\frac{1}{3}\times\frac{2}{3}+\frac{2}{3}\times\frac{1}{3}=\frac{4}{9}$.	1.5
3	$P=\frac{1}{2}\times\frac{4}{9}+\frac{1}{2}\times\frac{5}{18}=\frac{13}{36}$.	1.5

Q_6	Answers	Note
1.a	$\lim\limits_{x\to 0}\frac{f(x)}{x}=\lim\limits_{x\to 0}(\ln x-1)^2=+\infty$. At O (0 ; 0), the curve has (Oy) as a semi-tg	1
1.b	$\lim\limits_{x\to+\infty}f(x)=+\infty$. $\lim\limits_{x\to+\infty}\frac{f(x)}{x}=\lim\limits_{x\to+\infty}(\ln x-1)^2=+\infty$	1

<table>
<tr><td>2</td><td>

$f'(x) = (\ln x-1)^2+2(\ln x-1) = (\ln x)^2-1$. For $(\ln x)^2-1= 0 \Leftrightarrow (\ln x-1)(\ln x+1) = 0 \Leftrightarrow x = e$ or $x = \frac{1}{e}$.

<table>
<tr><td>x</td><td>0</td><td></td><td>$\frac{1}{e}$</td><td></td><td>e</td><td></td><td>$+\infty$</td></tr>
<tr><td>f′(x)</td><td></td><td>+</td><td>0</td><td>–</td><td>0</td><td>+</td><td></td></tr>
<tr><td>f(x)</td><td>0</td><td>↗</td><td>$\frac{4}{e}\approx 1.4$</td><td>↘</td><td>0</td><td>↗</td><td>$+\infty$</td></tr>
</table>

</td><td>2</td></tr>
<tr><td>3</td><td>

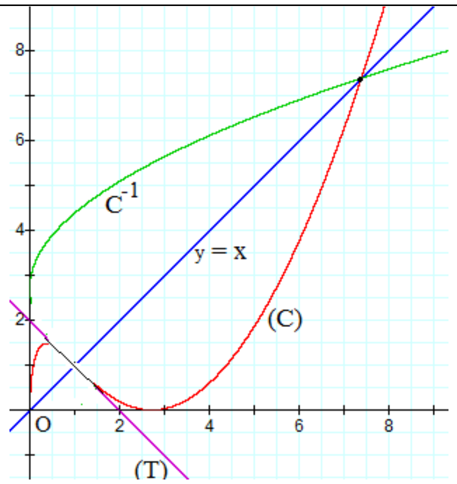
$f''(x) = 2\frac{\ln x}{x}$.
if $0 < x < 1$, then $\ln x < 0$, and $f''(x) < 0$.
if $x > 1$, $\ln x > 0$, and $f''(x) > 0$.
Since $f''(1)=0$ and f''(x) changes its sign then I (1 ; 1) is the inflection point of (C).
equation of (T) : $y - y_I = y'_I(x - x_I) \Rightarrow$ (T) : $y = -x+2$.

</td><td>1.5</td></tr>
<tr><td>4</td><td>

$x(\ln x-1)^2= x$; $x = 0$ or $(\ln x - 1)^2 - 1 = 0$, so $(\ln x-2)(\ln x)=0$
$\Rightarrow \begin{cases}\ln x = 0 \Rightarrow x = 1\\ \ln x = 2 \Rightarrow x = e^2\end{cases}$ hence J $(e^2 ; e^2)$.

</td><td>1</td></tr>
<tr><td>5</td><td>

(C) has, at $+\infty$, an asymptotic direction parallel to y′oy.



</td><td>2</td></tr>
<tr><td>6.a</td><td>

$F'(x) = x\left[(\ln x)^2 - 3\ln x + \frac{5}{2}\right] + \frac{x^2}{2}\left[2.\frac{1}{x}\ln x - \frac{3}{x}\right] = x\left[(\ln x - 1)^2\right] = f(x)$.

</td><td>1.5</td></tr>
<tr><td>6.b</td><td>

$A=\int_1^e f(x)dx - \int_1^2 (-x+2)dx = \left[\frac{x^2}{2}\left[(\ln x)^2 - 3\ln x + \frac{5}{2}\right]\right]_1^e + \left[\frac{x^2}{2} - 2x\right]_1^2 = \frac{e^2-7}{4}u^2$.

</td><td>1</td></tr>
<tr><td>7</td><td>

On [e ; + ∞[, f is continuous and strictly increasing, thus f has an inverse function g.
Curve (C^{-1}), see figure.

</td><td>1</td></tr>
<tr><td>8.a</td><td>

$x(\ln x - 1)^2 = m x$, $m > 0$, so $x[(\ln x - 1)^2 - m] = 0$, $x = 0$ or
$(\ln x - 1)^2 - m = 0$, $(\ln x - 1 - \sqrt{m})(\ln x - 1 + \sqrt{m}) = 0 \Rightarrow x = e^{1+\sqrt{m}}$ or $x = e^{1-\sqrt{m}}$.
$x_M = e^{1+\sqrt{m}} \Rightarrow y_M = m.e^{1+\sqrt{m}}$ and $x_{M'} = e^{1-\sqrt{m}} \Rightarrow y_{M'} = m.e^{1-\sqrt{m}}$.

</td><td>1</td></tr>
<tr><td>8b</td><td>

$x_P = e$ and $y_P = m.e$, $OP^2 = x_P{}^2 + y_P{}^2 = e^2(1+m^2)$
$\overrightarrow{OM}\cdot\overrightarrow{OM'} = e^{1+\sqrt{m}} \times e^{1-\sqrt{m}} + m^2 e^{1+\sqrt{m}} \times e^{1-\sqrt{m}} = e^2 + m^2 e^2 = OP^2$.

</td><td>1</td></tr>
</table>