امتحانات الشهادة الثانوية العامة
فرع العلوم العامة

الدورة العادية للعام 2013

الخميس 27 حزيران 2013	مسابقة في مادة الرياضيات	الاسم:
عدد المسائل: ست	المدة: أربع ساعات	الرقم:

ملاحظة: يُسمح باستعمال آلة حاسبة غير قابلة للبرمجة أو إختزان المعلومات أو رسم البيانات.
يستطيع المُرشّح الإجابة بالترتيب الذي يناسبه (دون الإلتزام بترتيب المسائل الوارد في المسابقة).

I- (2points)

Answer each of the following statements by true or false and justify the answer:

1) The points A, B and C with respective affixes $z_A = 2$, $z_B = 2e^{i\left(\frac{2\pi}{3}\right)}$ and $z_C = 2e^{i\left(-\frac{2\pi}{3}\right)}$ are the three vertices of an equilateral triangle.

2) For all nonzero natural numbers n, $Z = \dfrac{\left(1+i\sqrt{3}\right)^n - \left(1-i\sqrt{3}\right)^n}{2}$ is real.

3) For all real numbers x in the interval $]-1,0[$; $e^{|\ln(x+1)|} = x+1$.

4) For every real number b, the equation lnx = -2x + b has a unique solution in the interval $]0,+\infty[$.

II - (2 points)

In the space referred to a direct orthonormal system $(O; \vec{i}, \vec{j}, \vec{k})$, consider the two points A (1; 1; 1) and B (5 ; 2 ; 0) .

(P) and (P') are two planes with respective equations (P): x + 2y – 2z + 3 = 0 ; (P′): 2x + y + 2z = 0 .
Denote by (d) the line of intersection of (P) and (P').

1) Verify that a system of parametric equations of (d) is: $\begin{cases} x = -2t + 1 \\ y = 2t - 2 \\ z = t \end{cases}$ where t is a real parameter.

2) a- Show that the two planes (P) and (P′) are perpendicular.
   b- Calculate the respective distances from B to the planes (P) and (P') and calculate the distance from B to the line (d).

3) a-Determine an equation of the plane (Q) formed by the point B and the line (d).
   b- Prove that (d) and (AB) are skew.

4) a- Calculate the coordinates of E, the point of intersection of plane (P) and the line (AB).
   b- Show that the points A and B are located on the same side with respect to the plane (P).

III- (3 points)

The plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$. x and y are real numbers such that $y \neq 0$.

To every point M with affix $z = x + iy$, associate the point M′ with affix z′ so that $z' = z^3 + z$.

1) a- Prove that $(z-\overline{z})(z^2 + z\overline{z} + \overline{z}^2 + 1) = z' - \overline{z'}$ where $\overline{z}$ and $\overline{z'}$ are the respective conjugates of z and z′.

b- If z' is real, justify that $(z-\overline{z})(z^2 + z\overline{z} + \overline{z}^2 + 1) = = 0$.

c- Deduce that if z' is real, then the point M moves on the hyperbola (H) with equation $3x^2 - y^2 + 1 = 0$.

2) a- Determine the vertices of (H) as well as its asymptotes.

b- Determine one of the foci of (H) and its associated directrix.

c- Draw (H).

3) Let I be the point on (H) with abscissa 1 and positive ordinate.

a- Write an equation of (T), the tangent at I to (H).

b- The line (T) intersects the asymptotes of (H) at E and G. Prove that I is the midpoint of [EG].

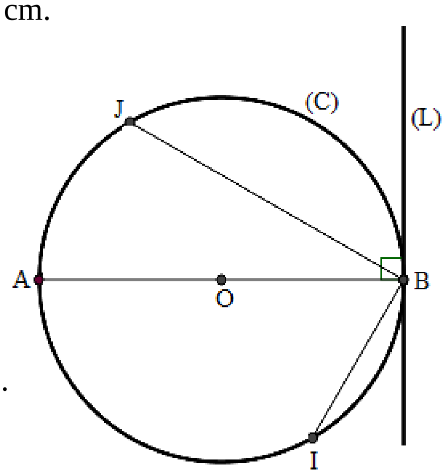
IV- (3 points)

In an oriented plane, consider a circle (C) with center O and radius 2 cm.

[AB] is a diameter of (C).

I and J are two points of (C) so that $\left(\overrightarrow{BI},\overrightarrow{BA}\right) = -\frac{\pi}{3}\,[2\pi]$

and $\left(\overrightarrow{BA},\overrightarrow{BJ}\right) = -\frac{\pi}{6}\,[2\pi]$.



The line (L) is tangent at B to the circle (C).

Consider the direct similitude S with center B that transforms I onto J.

1) Determine an angle of S and verify that its ratio k is $\sqrt{3}$.

2) a- Show that (AJ) is the image of the line (AI) under S.

b- Find the image of the line (AB) under S.

c- Deduce S (A) and then find S (J).

3) Let (C′) be the image of (C) under S. Determine (C′) and calculate its area.

4) Let $S_2 = S \circ S$, $S_3 = S \circ S \circ S$,……………, $S_n = \underbrace{S \circ S \circ S \circ \ldots \circ S}_{\text{n times}}$.

n being a natural number $(n \geq 2)$.

a- Verify that S o S is a dilation whose center and ratio are to be determined.

b- Determine, in terms of n, the ratio and an angle of the similitude S_n .

c- Find the values of n for which S_n is a dilation.

V- (3 points)

An urn contains five red balls and five green balls.

Three balls are selected, simultaneously and at random, from the urn.

Consider the following events:

- E: « The three selected balls are red ».
- F: « Among the three selected balls, there are exactly two red balls ».
- S: « Among the three selected balls, there is at most one red ball ».

1) Calculate the probabilities P (E), P (F) and P (S).

2) In this question, a game runs in the following way:

A player selects, simultaneously and at random, three balls from the urn.

- If the event S occurs, the player gains **nothing** and the game ends.
- If one of the two events E or F occurs; then the player selects a new ball from the seven remaining balls:

  - If this selected ball is green, the player gains 10 points.

  - Otherwise, the player gains 2 points.

Consider the event T: « The player gains ten points ».

a- Calculate P (T/E) and P (T/F).

b- Prove that the probability P (T) is equal to $\frac{25}{84}$.

c- The player gains 10 points. What is the probability that the three selected balls are red?

d- Denote by X the random variable equal to the score of points of the player.

Determine the probability distribution of X and calculate its expected value E(X).

VI – (7 points)

A-

Consider the differential equation (E): $y''-4y'+4y=4x-4$ where y is a function of x.

Let $y=z+x$.

1) Find the differential equation (E') satisfied by z.

2) Solve (E') then deduce the general solution of (E).

3) Determine the particular solution of (E) whose representative curve in an orthonormal system $\left(O\ ;\vec{i},\vec{j}\right)$ has at the point G with abscissa 0 a tangent with equation $y=x-1$.

B-

Let g be the function defined on $\mathbb{R}$ as $g(x) = 4xe^{2x}+1$.

1) Determine $g'(x)$ and set up the table of variations of g.

2) Deduce the sign of g(x).

C-

In what follows, let f be the function defined on $\mathbb{R}$ as $f(x)=x+(2x-1)e^{2x}$ and denote by (C) its representative curve in the system $\left(O\ ;\vec{i},\vec{j}\right)$.

1) a- Show that the line (d) with equation $y = x$ is an asymptote to (C),

b- Study, according to the values of x, the relative positions of (C) and (d) and specify the coordinates of A, their point of intersection.

c- Determine $\lim\limits_{x\to+\infty} f(x)$.

2) a- Verify that $f'(x) = g(x)$ and set up the table of variations of f.

b- Prove that (C) intersects the x- axis at a unique point K with abscissa α then verify that $0.40 < \alpha < 0.41$.

c- Draw (C).

3) The function f has over $\mathbb{R}$ an inverse function h. Denote by (H) the representative curve of h.

a- Show that A is on (H) and find an equation of the tangent to (H) at A.

b- Draw (H) in the same system as (C).

c- Calculate $\int (2x-1)e^{2x}dx$ and deduce the area S of the region bounded by (H), the x- axis and the line (d).

4) Let n be a natural number such that $n \geq 2$.

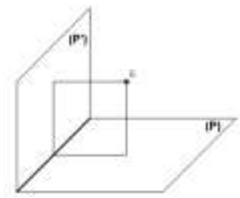
a- Use mathematical induction to prove that $f^{(n)}(x) = 2^n[2x + n - 1]\, e^{2x}$, where $f^{(n)}$ is the n^{th} derivative of the function f.

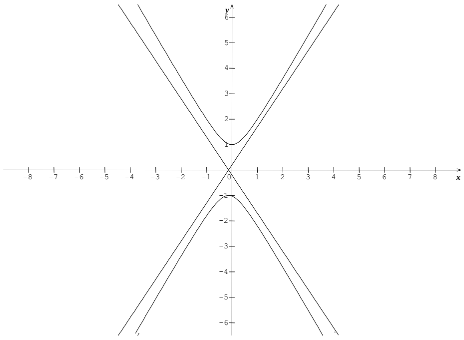
b- Study the sense of variations of the sequence (U_n) whose general term is $U_n = f^{(n)}(0)$.

c- Show that the sequence (U_n) is not convergent.

Bareme – Math – S.G – 2013

Q_1	Answers	M
1	The 3 complex numbers have the same modulus 2, then the points A, B and C are on the same circle with center and radius 2. Also, $\widehat{AB} = \widehat{BC} = \widehat{CA} = \frac{2\pi}{3}$ thus the points A, B and C are the vertices of an equilateral triangle . **True**	1
2	$(1 - i\sqrt{3})^n$ is the conjugate of $(1 + i\sqrt{3})^n$, thus Z is pure imaginary. **False**	1
3	For $-1 < x < 0$ $0 < x+1 < 1$, $\ln(x+1) < 0$ then $f(x) = e^{-\ln(x+1)} = \frac{1}{x+1}$ **.False**	1
4	Considering the function f defined over]0 ; +∞ [by f(x) = lnx + x – b. $f'(x) = \frac{1}{x} + 1$ f '(x) > 0 then f is strictly increasing with $\lim_{x\to 0} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x\left(\frac{\ln x}{x} + 1 - \frac{b}{x}\right) = +\infty$, thus the equation f(x) = 0 has a unique solution in □ . **True.**	1

Q_2	Answers	M
1	M is a variable point on (d), hence $x_M = -2t+1$, $y_M = 2t-2$ and $z_M = t$. $x_M + 2y_M - 2z_M + 3$ = –2t + 1 + 4t – 4 –2t + 3 = 0 ; then (d) lies in (P). $2x_M + y_M + 2z_M$ = – 4t + 2 + 2t – 2 + 2t = 0 ; then (d) lies in (P'). Hence, (d) is the line of intersection of (P) and (P′).	0.5
2a	$\overrightarrow{n_P}$ (1; 2; –2), $\overrightarrow{n_{P'}}$ (2; 1; 2). $\overrightarrow{n_P} \cdot \overrightarrow{n_{P'}}$ = 2 +2 – 4 = 0; thus (P) and (P′) are perpendicular.	0.5
2 b	d_1 = d (B →(P)) = $\frac{\lvert 5+4+3\rvert}{3}$ = 4. d_2 = d (B →(P')) = $\frac{\lvert 10+2\rvert}{3}$ = 4. (P) and (P′) are perpendicular. $[d(B\to(d))]^2 = d_1^2 + d_2^2 = 32$; d (B→(d)) = $4\sqrt{2}$. **Or** by direct calculation. 	1
3 a	For z = 0; G (1; –2; 0) ∈(d), $\overrightarrow{v_d}$ (–2; 2; 1) and $\overrightarrow{GB}(4;4;0)$. Let M(x; y; z) be any point on (Q), then $\overrightarrow{GM} \cdot (\overrightarrow{GB} \wedge \overrightarrow{V_d}) = \begin{vmatrix} x-1 & y+2 & z \\ 4 & 4 & 0 \\ -2 & 2 & 1 \end{vmatrix} = 0$. An equation of (Q) is x – y + 4z – 3 = 0.	0.5
3 b	A(1;1;1), B(5;2;0), G(1;–2;0)∈(d) , F(–1;0;1) ∈(d) $\overrightarrow{AB} . (\overrightarrow{AG} \wedge \overrightarrow{GF}) = \begin{vmatrix} 4 & 1 & -1 \\ 0 & -3 & -1 \\ -2 & 2 & 1 \end{vmatrix}$ = 4 ≠ 0; (AB) and (d) are skew. **OR** If (AB) and (d) were coplanar, then A ∈ (Q) but $x_A - y_B + 4z_A - 3 \neq 0$ and B ∈ (Q) and (d) ⊂ (Q). Thus, (AB) and (d) are skew.	0.5
4 a	E∈(AB) ; E (4k +1 ; k + 1 ; – k + 1) ; E∈(P) then $x_E + 2y_E - 2z_E + 3 = 0$; Therefore k = $-\frac{1}{2}$ and E $(-1; \frac{1}{2}; \frac{3}{2})$.	0.5
4 b	$\overrightarrow{EA}(2; \frac{1}{2}; -\frac{1}{2})$ and $\overrightarrow{EB}(6; \frac{3}{2}; -\frac{3}{2})$; $\overrightarrow{EA} . \overrightarrow{EB} = 12 + \frac{3}{4} + \frac{3}{4} = \frac{27}{2} > 0$. Hence, the two points A and B are located in the same side with respect to (P). **OR** $\overrightarrow{EB} = 3\overrightarrow{EA}$.	0.5

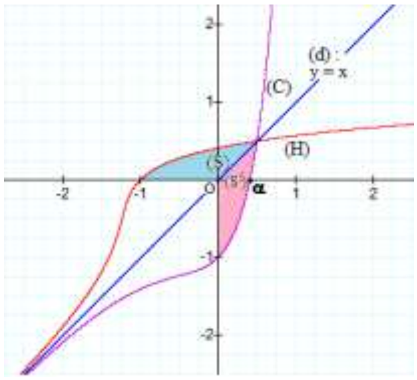
Q_3		M
1a	$(z-\bar{z})(z^2+z\bar{z}+\bar{z}^2+1)= z^3+z^2\bar{z}+z\bar{z}^2+z-\bar{z}z^2-z\bar{z}^2-\bar{z}^3-\bar{z} = z'-\bar{z}'$	0.5
1b	z' is a real ; $z'=\bar{z'}$ hence $z'-\bar{z'}=0$ then $(z-\bar{z})(z^2+z\bar{z}+\bar{z}^2+z+\bar{z}+1)=0$.	0.5
1c	z' is a real number, then $z-\bar{z}=0$: which gives $z=\bar{z}$, and y = 0 to be rejected. OR $z^2+z\bar{z}+\bar{z}^2+z+\bar{z}+1=0 \rightarrow 3x^2-y^2+2x+1=0$ with $y \neq 0$. Hence, M belongs to curve (H) with equation $3x^2-y^2+1=0$.	1
2 a	$3x^2-y^2+1=0 \rightarrow y^2-3x^2=1 \rightarrow y^2-\frac{x^2}{\frac{1}{3}}=1$. The vertices are A (0 ; 1) and A'(0 ;-1). The asymptotes have the equations : (L) : $y=-\sqrt{3}x$ and (L') : $y=\sqrt{3}x$	1
2 b	$c^2=\frac{4}{3}$ then $F(0,\frac{2\sqrt{3}}{3})$ and the associated directrix has an equation $y=\frac{\sqrt{3}}{2}$.	0.5
2c		0.5
3a	$3x^2-y^2+1=0$ gives $6x-2yy'=0$ thus, $y'=\frac{3x}{y}=\frac{3}{2}$. The equation of the tangent (T) is $y=\frac{3}{2}x+\frac{1}{2}$	1
3 b	E = (T) $\cap$ (L): $2x\sqrt{3}-3x=1$ then $x_E=\frac{1}{2\sqrt{3}-3}$. G = (T) $\cap$ (L') gives $x_G=\frac{1}{-2\sqrt{3}-3}$. $x_E+x_G=\frac{1}{2\sqrt{3}-3}+\frac{1}{-2\sqrt{3}-3}=2=2x_I$. And since E, G and I are collinear , then I is the midpoint of [EG].	1

Q4	Answers	M
1	S = sim (B; k;α): I $\longrightarrow$ J. $\alpha=(\overrightarrow{BI},\overrightarrow{BJ})=(\overrightarrow{BI},\overrightarrow{BA})+(\overrightarrow{BA},\overrightarrow{BJ})=-\frac{\pi}{3}-\frac{\pi}{6}=-\frac{\pi}{2}$ (mod2π). triangle IBJ is semi equilateral since $\widehat{IBJ}=\frac{\pi}{2}$ and $\widehat{JIB}=\frac{\pi}{3}$, then $k=\frac{BJ}{BI}=\tan\frac{\pi}{3}=\sqrt{3}$.	1
2 a	S ((AI)) is the line passing through the point J = S (I) and perpendicular to (AI), S ((AI)) = (AJ) since AJBI is a rectangle.	0.5
2 b	S ((AB)) is the line passing through the point B = S(B) and perpendicular to (AB), S ((AB)) = (L).	0.5
2 c	• $\left.\begin{array}{l} A \in (AI) \rightarrow S(A)=A' \in S(AI)=(AJ) \\ A \in (AB) \rightarrow S(A)=A' \in S(AB)=(L) \end{array}\right\}$ A' is the point of intersection of (AJ) and (L). • S (J) = J'. AIBJ is a direct rectangle therefore A'JBJ' is a direct rectangle.	1
3	• (C) is the circle with diameter [AB], S ((C)) = (C′) which is the circle with diameter S [AB] = [A'B]. • $A_{(C')}=k^2A_{(C)}=3\pi\times 2^2=12\pi u^2$.	1

4a	S ∘ S =Sim(B,3,- π) thus, S ∘ S= dilation (B,-3).	0.5
4b	S_n = sim (B ; ($\sqrt{3}$) ^{n} ; $-n\dfrac{\pi}{2}$).	0.5
4c	S_n is a dilation if and only if $-n\dfrac{\pi}{2} = k\pi$; then n = –2k (k<0); n is even.	1

Q5	Answers	M
1	$P(E)=\dfrac{C_5^3}{C_{10}^3}=\dfrac{1}{12}$. $P(F)=\dfrac{C_5^2\times C_5^1}{C_{10}^3}=\dfrac{5}{12}$. P(S)= P(3G or 1R and 2G)= $\dfrac{C_5^3}{C_{10}^3}+\dfrac{C_5^1.C_5^2}{C_{10}^3}=\dfrac{10}{120}+\dfrac{50}{120}=\dfrac{1}{2}$. **OR** p(S)=1-p(E)-P(F)=1- $\dfrac{1}{12}-\dfrac{5}{12}=\dfrac{1}{2}$.	1.5
2a	P(T/E)=P (1green ball/E) = $\dfrac{5}{7}$. P(T/F) = $\dfrac{4}{7}$	1
2b	$P(T)=P(T\cap E)+P(T\cap F)=P(T/E)\times P(E)+P(T/F)\times P(F)=\dfrac{5}{7}\times\dfrac{1}{12}+\dfrac{5}{12}\times\dfrac{4}{7}=\dfrac{25}{84}$.	1
2c	P(E/T)= $\dfrac{P(E\cap T)}{P(T)}=\dfrac{\frac{5}{7}\times\frac{1}{12}}{\frac{25}{84}}=\dfrac{1}{5}$.	1
2d	X (Ω) = {0 ; 2 ; 10}. P (X= 0) = P (S) = $\dfrac{1}{2}$. P (X=10) = P(T)= $\dfrac{25}{84}$. P (X=2) = 1-[P (Y= 0) + P (Y= 10)] = $\dfrac{17}{84}$. $E(X)=\sum x_i\times p_i=0\times\dfrac{1}{2}+2\times\dfrac{17}{84}+10\times\dfrac{25}{84}=\dfrac{284}{84}$. E(X) = $\dfrac{284}{84}$ =3.38.	1.5

Q_6	Answers	M
A1	y'=z'+1, y"=z" then z"-4z'-4+4z+4x=4x-4 thus, z"-4z'+4z=0.	0.5
A2	C.E. : r^2-4r+4=0. double root r=2 General solution of (E'): z= (ax+b)e^{2x} . Hence, y= (ax+b)e^{2x}+x	1
A3	f(0)=-1 anf f'(0)=1 then b=-1. f '(x)=1+ae^{2x} +2(ax+b)e^{2x} so f '(0)= 1+a+2b=1 thus a=2. Hence, f(x)= x+(2x-1)e^{2x} .	1
B1	g ′(x) = 4 e^{2x} (1 + 2x). x: $-\infty$; $-\dfrac{1}{2}$; $+\infty$ g′(x): – 0 + g(x): 1 ↘ $1-\dfrac{2}{e}$ ↗ $+\infty$	1
B2	g(x) has a positive minimum hence g(x) > 0 for all x.	0.5
C1a	$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}(x+(2x-1)e^{2x})=\lim\limits_{x\to-\infty}(x+2xe^{2x}-e^{2x})=-\infty+0+0=-\infty$. $\lim\limits_{x\to-\infty}(f(x)-x)=\lim\limits_{x\to-\infty}(2x-1)e^{2x}=0$. Hence the line (d): y = x is an asymptote to (C) at $-\infty$.	0.5
C1b	f(x) – y = $(2x-1)e^{2x}$ = 0 for x = $\dfrac{1}{2}$. Hence for x = $\dfrac{1}{2}$, (C) and (d) intersect at point A ($\dfrac{1}{2}$; $\dfrac{1}{2}$) ; If x < $\dfrac{1}{2}$ then (C) is under (d); and if x > $\dfrac{1}{2}$ then (C) is above (d).	1
C1c	$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}(x+(2x-1)e^{2x})=+\infty$.	0.5

<table>
<tr><td>C2a</td><td>$f'(x)=1+2e^{2x}+2e^{2x}(2x-1)= g(x)$.
<table><tr><td>x</td><td>$-\infty$ $+\infty$</td></tr><tr><td>$f'(x)$</td><td>+</td></tr><tr><td>f(x)</td><td>$-\infty$ ↗ $+\infty$</td></tr></table></td><td>1</td></tr>
<tr><td>C2b</td><td>Over ℝ , f is continuous and strictly increasing from $-\infty$ to $+\infty$ then f(x) = 0 has one and unique solution. Moreover, $f(0.4) \times f(0.5) = -0.045 \times 0.5 < 0$ so $0.4 < \alpha < 0.5$.</td><td>1</td></tr>
<tr><td>C2c</td><td>$\lim\limits_{x\to+\infty}\frac{f(x)}{x} = \lim\limits_{x\to+\infty}[1+\frac{2x-1}{x}e^{2x}]=+\infty$.
Hence (C) has at $+\infty$ an asymptotic direction parallel to the y-axis.

</td><td>1</td></tr>
<tr><td>C3a</td><td>$A=(C)\cap(d)$; so A is its own symmetric with respect to (d) hence A belongs to (H).
$y=\frac{1}{f'\left(\frac{1}{2}\right)}\left(x-\frac{1}{2}\right)+\frac{1}{2}=\frac{1}{2e+1}\left(x-\frac{1}{2}\right)+\frac{1}{2}$.</td><td>1</td></tr>
<tr><td>C3b</td><td>(H) is the symmetric of (C) with respect to (d).(See figure)</td><td>0.5</td></tr>
<tr><td>C3c</td><td>Let $u = 2x-1$ and $v' = e^{2x}$ so $u' = 2$ and $v=\frac{1}{2}e^{2x}$ which gives :
$\int(2x-1)e^{2x}dx=\frac{1}{2}(2x-1)e^{2x}-\int e^{2x}dx=(x-1)e^{2x}+c$.
Using symmetry, the required area S, is that area S′ of the region bounded by (C), line (d), and the y – axis.
$S'=\int_0^{0.5}(x-f(x))\,dx=\int_0^{0.5}(1-2x)e^{2x}\,dx=\left[(1-x)e^{2x}\right]_0^{0,5}=\frac{e-2}{2}$. Hence $S = S' = \frac{e-2}{2}$ u^2.</td><td>1.5</td></tr>
<tr><td>C4a</td><td>Let $a_n = 2^n[2x+n-1]e^{2x}$. For n = 2; $f''(x) = 4(e^{2x}+2xe^{2x}) = 4(2x+1)e^{2x}$.
$a_2 = 4(2x+1)e^{2x}$. True for n = 2.
Assume that $f^{(n)}(x) = a_n$ and prove that $f^{(n+1)}(x) = a_{n+1}$.
$f^{(n+1)}(x) = [f^{(n)}]'(x) = 2^n[2+2(2x+n-1)]e^{2x} = 2^n[4x+2n]e^{2x} = 2^{n+1}[2x+n]e^{2x} = a_{n+1}$.
True for all $n \geq 2$.</td><td>1</td></tr>
<tr><td>C4b</td><td>$U_n = 2^n(n-1)$;
$U_{n+1} - U_n = 2^{n+1}(n) - 2^n(n-1) = 2^n(2n-n+1) = 2^n(n+1) > 0$; (U_n) is strictly increasing.</td><td>0.5</td></tr>
<tr><td>C4c</td><td>2>1 then $\lim\limits_{n\to\infty}2^n=+\infty$ thus, $\lim\limits_{n\to\infty}U_n=\lim\limits_{n\to\infty}2^n(n-1)=+\infty$, hence (U_n) is not convergent.</td><td>0.5</td></tr>
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